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THE

PRINCIPLES

OF

ALGEBRA:

OR

THE TRUE THEORY OF EQUATIONS

ESTABLISHED ON

MATHEMATICAL DEMONSTRATION.

PART THE SECOND.

BY

WILLIAM FREND.

LONDON:

SOLD BY

G. G. AND J. ROBINSON, PATERNOSTER-ROW; WHITE, FLEET-STREET; RIDGEWAY, YORK-STREET, ST.-JAMES'S-SQUARE; AND DEIGHTON, CAMBRIDGE.

1799.

TO THE

REV. ROBERT TYRWHITT, M. A.

THIS VOLUME,

INDEBTED TO HIM, IN GREAT MEASURE,

FOR ITS EXISTENCE,

IS DEDICATED

BY THE AUTHOUR.

PREFACE.

FROM the age of Vieta, the father, to this of Maseres, the restorer of Algebra, many men of the greatest abilities have employed themselves in the pursuit of an idle hypothesis, and have laid down rules not founded in truth, nor of any sort of use in a science admitting in every step the plainest principles of reasoning. If the name of Sir Isaac Newton appears in this list, the number of the advocates for errour must be considerable. It is, however, to be recollected, that for a much longer period, men scarcely inferiour to Newton in genius, and his equals probably in industry, maintained a variety of positions in philosophy, which were overthrown by a more accurate investigation of nature; and, if the name of Ptolemy can no longer support his epicycles, nor that of Des Cartes his vortices, Newton's dereliction of the principles of reasoning cannot establish the fallacious notion, that every equation has as many roots as it has dimensions.

This

This notion of Newton and others is founded on precipitation. Inſtead of a patient examination of the ſubject, an hypotheſis, which accounts for many appearances, is formed: where it fails unintelligible terms are uſed: in theſe terms indolence acquieſces: much time is waſted on a jargon which has the appearance of ſcience, and real knowledge is retarded. Thus volumes upon volumes have been written on the ſtupid dreams of Athanaſius, and on the impoſſible roots of an equation of *n* dimenſions.

It was very early perceived that equations in certain forms admitted ſeveral roots, and that by multiplying together complex terms in certain forms, terms ſimilar to theſe equations might be produced. Hence, from the natural deſire of mankind to come as ſoon as poſſible to a general concluſion, the thought ſuggeſted itſelf, that all equations might be produced by a multiplication of ſimilar complex terms, and that the analogy which had been obſerved between the coparts and roots of certain equations might be extended in general to all equations. The idea thus advanced was without ſufficient attention to the natural difference, which ſubſiſts in the different order of equations, adopted. It is true, that all the equations which have more than one term may be formed by the multiplication of two or more complex terms; but it depends

depends upon the form of an equation, whether it admits of a multiplication by a number of complex terms in the forms propoſed equal to the higheſt index of the given equation. Thus let

$$x^2 + a x = k.$$

Let the root of the above equation be equal to b, then $\overline{x - b} \times \overline{x + a + b}$ is equal to $x^2 + a x - k$; but a and b are added to x and conſequently $a + b$ cannot be a root of the given equation. In the ſame manner let

$$x^m + a x^n + b x^o + c x^p + d x^q \ldots\ldots = k.$$

If l is one root, then $x - l$ multiplied into ſome complex term may form the equation; but this complex term is equal to an aſſignable number, and the number ſubſtituted for x in this complex term, which makes it equal to the aſſignable number, cannot be the root of the propoſed equation. With true reaſoners this would have been ſufficient to make them reject all thoughts of producing every equation by the multiplication of a number of complex terms equal to the higheſt index in the given equation; but unfortunately a fiction was introduced, and ſcience was placed upon a level with the chicanery of courts of law in ſome countries, where the firſt ſtep to obtain redreſs for certain injuries commences with a falſehood. Though it is certain that in the inſtance above, where $\overline{x + a} \times \overline{x + a + b}$ produces the given

equation, $a + b$ is added to x, yet by a ſtrange fiction it was to be ſuppoſed that $a + b$ is taken from x, and that $a + b$ is equal to a number c, to be called either a negative or an impoſſible number. The nature of theſe fictitious numbers now became an object of inquiry, and, inſtead of ſearching after the roots of an equation, even grave men, imitating the philoſophers of a well-known region, who were extracting ſun-beams from cucumbers, waſted the midnight oil juſt as profitably in ſettling the rights and privileges of impoſſible quantities.

The fruits of their lucubrations I ſucked in with the firſt milk of alma mater; and, if the good old lady had not driven me from my books and from ſtudies of a much ſuperiour nature, it is not improbable that I might ſtill have looked at theſe ſubjects through her ſpectacles. It is of little conſequence to the publick how, or why, I came back to my former ſtudies; ſuffice it, that being no longer of an age nor of a diſpoſition to take for granted an old wive's tale, I ſtarted at the poſtulate to make a load of corn out of a buſhel. So ſtrange a demand now excited my inquiry into the principles of a ſcience whoſe objects are all of a determinate nature, and after conſiderable reſearches I diſcovered the direct demonſtrations which this volume contains. A ſpecimen of them I gave

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laſt ſummer in a Letter * to the Vice Chanceller of the Univerſity of Cambridge on the vacancy of a profeſſerſhip, which was filled up by one whoſe pretenſions to be a candidate were, in the opinion of moſt people, againſt the ſpirit, and, in that of many, againſt the letter of the ſtatute. In my opinion his election was againſt both; but, if he does any thing worthy of that profeſſerſhip, I ſhall not be very ſcrupulous about his right of obtaining it, and ſhall not, moſt aſſuredly, in theſe times, diſturb him in the poſſeſſion of it.

In this work the number of roots in an equation is determined, not by a fiction, but on certain and undeniable principles; a direct demonſtration is given of the relation of the coparts and roots in ſeveral equations, and the mode pointed out for all equations; the limiting equations are diſcovered by a ſimple principle which appears to me capable of great extenſion. In this and ſimilar queſtions fluxions are generally applied, that is, the new and unneceſſary quality of velocity is introduced in queſtions depending on mere addition and ſubtraction. My principle is ſimply this. Let *a* and *b* be certain numbers,

* A Letter to the Vice Chanceller of the Univerſity of Cambridge, by William Frend, Candidate for the Lucaſian Profeſſerſhip. Price 6d. White, Fleet-ſtreet; and Flower and Deighton, Cambridge.

z and y variable numbers; then if

$$a = b + Az - By + Cz^2 - Dy^2 + Ez^3 - Fy^3 \ldots$$

the difference between the variable terms added and thofe fubtracted muft be equal either to a given number or to nothing, otherwife a would be a variable number. Now the difference cannot in my demonftrations be in any cafe a given number, for both z and y may be taken lefs than any affignable number. Confequently, if this difference was any determinate number, when z and y were each equal to fome given numbers, this difference, when z and y were both made lefs than any affignable number, would be lefs than the preceding determined number, that is, the difference would be a variable number which might be expreffed by v, and then a would be equal to $b \pm v$, which is impoffible, for a can be equal to only one number. The advantage of this demonftration is, that z and y may be tried in any given equation, and the refult made evident to the learner in every inftance. If $12x - x^2 = 20$, then the limiting equation $12 = 2x$ or $x = 6$, and if z is made equal to unity then y is equal to unity, and 6 ± 1 being fubftituted for x in the given equation produce the fame refults; if $z = 2$, then $y = 2$, and fo on *.

* See Pages 26 — 29, 67 — 69.

I

I have alſo given in this work a conſiderable number of examples, particularly from Raphſon; for I have found by experience that a learner, who is ſatisfied with general expreſſions and general demonſtrations without trying the reſult in practice ſeldom acquires any ſolid knowledge, and accordingly it is a great point with me in my preſent mode of teaching to accompany in every inſtant the theory with practice. The methods adopted by ſeveral eminent mathematicians in the ſolution of equations are pointed out with their merits and defects: and here I ought particularly to obſerve that the demonſtration given in pages 13 and 14 is taken chiefly from Dr. Hutton's eſſays; and, though I cannot attribute to him the whole merit of that method, I can eaſily conceive that he did not derive any aſſiſtance from others; and his great merits will not ſuffer by the juſt tribute of applauſe being beſtowed on De Lagny his predeceſſer in this diſcovery. By the compariſon of my mode of dividers with that of Dr. Hutton's by double poſition, pages 96, 97, I do not by any means conſider the queſtion as thoroughly decided, for though I have no doubt in my own mind on the preference of my mode to his in all the forms of equations, which are the ſubject of this work, I muſt leave to the learner and to farther inquiry on my own part thoſe equations, where the unknown term is un-

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der a radical band, and where at present I conjecture only that my mode has the advantage.

It might be proper here to acknowledge again the egregious oversight committed in the first part of my Algebra, page 213, if I had not already done it in the Monthly Magazine, and to my mistake the publick is indebted for an appendix, in which Baron Maseres has given a complete examination of Cardan's rule, and a most accurate comparison of Ferrari's solution, with that of Raphson, by his method of approach. I may presume also to hope that the tutours of the University of Cambridge will recommend to their pupils the five last pages of that appendix to rectify their notions on the fiction of negative and impossible quantities; and, if they dilate on the concluding paragraph, they will probably make an interesting comparison between the similar fictions of modern algebraists and antient divines.

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N° 9, Inner Temple-lane.

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From

CONTENTS.

PRINCIPLES OF ALGEBRA.

PART THE SECOND.

CHAP. I.

ON EQUATIONS IN GENERAL.

FROM the former part of this work it appears, that ſome equations have more roots than one: and as the diſcovery of the roots of equations is a principal part of Algebra, a perſon cannot be a proficient in the ſcience, unleſs he ſhall be able, on certain grounds, to determine how many roots an equation in any form propoſed to him can admit. The Appendix has alſo ſhewn with what difficulty the ſolution of ſeveral equations muſt be attended; and, as this difficulty increaſes with the number of terms on the unknown ſide of an equation, it will be of uſe to examine the means of ſhortening this labour, and of confining the firſt approaches to the roots within ſome aſſignable limits. Hence this part of the principles of Algebra will be confined chiefly to theſe two enquiries; into

the number of roots in any equation, and into the limits within which these roots are confined; and though the forms of equations are more numerous than the leaves of autumn in Vallombrosa; and out of the myriads of these forms, the ingenuity of man has not hitherto discovered a general expression for the root in one form out of a million; yet, by certain methods, the root of any equation which can be expressed by decimal arithmetick, will, by a degree of labour proportional to the number of figures in the root and the number of terms in the equation, be discovered.

As the forms of equations are so numerous, it will be expedient to adopt some mode of dividing them into classes: and these classes should be, if possible, of such a nature, that a general property may go through the forms of every class. There is, first, a general division of equations according to the nature of their unknown terms; for they may admit either of only one unknown number with its powers, or of several unknown numbers with their powers; or the index of the number in any term may be an unknown number. Thus the equation:

$x^{n} \pm ax^{n-1} \mp bx^{n-2} \pm cx^{n-3} \ldots\ldots = k$ has on one side only x and its powers.

The equation

$x^{n} \pm vx^{n-1} \mp v^{2}yx^{n-2} \pm v^{3}y^{2}zx^{n-3} \mp \ldots\ldots\ldots = k$ has several unknown numbers v, y, z, &c. with their powers.

The equation

$a^{x} \pm b a^{x-1} y + c a^{x-2} \pm d a^{x-3} \ldots\ldots\ldots = k$ has not only simple unknown numbers with their powers; but the indexes also of several numbers are unknown.

Of these equations, the first only will make the principal part of this enquiry; namely, those forms in which a simple unknown number or its powers are involved; and equations of this kind will be classed according to the number of terms on the unknown side. Thus all equations will be considered of the first class, which have only one unknown term, and consequently only one root; and these equations come under the form $x^{n} = k$.

Equations of the second class have only two unknown terms, and cannot have more than two roots, and are reducible to one or the other of the three forms:

$$x^{m+n} + a x^{n} = k.$$
$$x^{m+n} - a x^{n} = k.$$
$$a x^{n} - x^{m+n} = k.$$

Equations of the third class have three unknown terms; of the fourth class, four unknown terms, and so of the higher classes. One general rule pervades all equations: namely, that no equation in any class can have more roots than it has unknown terms; and in the forms of each class the number of roots depends partly upon the coparts of the unknown terms, and partly upon the

changes of the marks of adding or taking away. In equations whose unknown terms are ranged according to the order of their powers, there can be only one root, if there is only one or no change of the marks; and in equations, not ranged according to the order of the powers, there will be only one root, if there is a certain relation between the known term of the coparts of the unknown terms. Thus equations in these forms

$$x^{n} + a x^{n-1} + b x^{n-2} + \ldots + \ldots + \ldots + = k$$

$$x^{n} - a x^{n-1} - b x^{n-2} - \ldots - \ldots - \ldots - = k,$$

can have only one root.

CHAP. II.

EQUATIONS OF THE FIRST CLASS.

Equations of the first class, after bringing the known terms to one side, and the unknown to the other, and freeing the unknown term of its known parts, are of this form:

$$x^{n} = k.$$

To find therefore the number x, the n^{th} root of k is to be found.

If $n = 2$, the second root may be found by the rules laid down in the first Part (page 92, and the following pages).

If

If $n = 3$, the third root may be found by the rules laid down in the first Part, (page 96, and the following).

If $n = 4$, the fourth root may be found upon similar principles, and so on for any number n.

But these methods of finding the root are very troublesome, especially in the higher powers; and therefore, when the root is contained within six or seven figures, it is discovered in the easiest manner by the logarithmical tables; or, if a number is required which shall approach as nearly as possible to the root of a given number, the first six or seven figures are discovered in the easiest manner by logarithms. Thus, let

$$x^2 = 2.$$

$$\therefore 2,\ \text{Log. } x = \text{Log. } 2 = ,30103$$

$$\therefore \text{Log. } x = ,150515$$

$$\text{Log. } 1,41421 = ,1505139.$$

Diff. between Log. 1,414210 and Log. 1,414220 = ,0000031.

$$\frac{\text{Diff.}}{10} = ,00000031.$$

$$\therefore \frac{3.\ \text{Diff.}}{10} = ,00000093.$$

$$\therefore \text{Log. } 1,414213 = \left.\begin{array}{r} ,1505139 + \\ + ,00000093 \end{array}\right\} = ,15051483.$$

$$\text{Log. } 1,4142130 = ,15051483$$

$$\text{Log. } 1,4142140 = ,15051514.$$

$$\therefore \text{Diff.} = ,00000031.$$

∴

$$\therefore \frac{\text{Diff.}}{10} = ,00000031$$

$$\therefore \frac{5 \text{ Diff.}}{10} = ,000000155.$$

$$\text{Add Log. } 1,4142130 = ,15051483$$

$$\therefore \text{Log. } 1,4142135 = ,150514985.$$

$$\text{Log. } 1,4142135 = ,150514985$$

$$\text{Log. } 1,4142136 = ,150515016$$

$$\therefore \text{Diff.} = ,000000031$$

$$\frac{\text{Diff.}}{10} = ,0000000031$$

$$\therefore \frac{6.\ \text{Diff.}}{10} = ,0000000186$$

$$\text{Add Log. } 1,4142135 = ,150514985.$$

$$\therefore \text{Log. } 1,41421356 = ,1505150036;$$

but ,1505150036 is greater than ,150515; therefore 1,41421356 is by the tables greater than the second root of two, which is not true, for the second root of two is greater than 1,41421356, and the error is owing to the defect in the tables, which are calculated only to seven places of decimals. But the figures 1,4142135 are right.

$$\text{Let } x^3 = 37945.$$

$$\therefore 3 \text{ Log. } x = \text{Log. } 37945 = 4,5791546$$

$$\therefore \text{Log. } x = 1,5263848 66 \ldots.$$

$$\therefore x = 33,603522 \ldots.$$

The first seven figures are true.

Let

$$\text{Let } x^4 = 2741583974.$$

$$\text{Suppose } x = 10y.$$

$$\therefore 10^4 y^4 = 2741583974$$
$$\therefore y^4 = 274158{,}3974$$
$$\therefore 4 \text{ Log. } y = \text{Log. } 274158{,}3 \text{ nearly} = 5{,}43800168$$
$$\therefore \text{Log. } y = 1{,}35950042$$
$$\therefore y = 22{,}882330 \ldots$$
$$\therefore x = 228{,}82330 \ldots$$

The seven first figures are true.

$$\text{Let } x^5 = 23278345598 73.$$

$$\text{Suppose } 100y = x$$
$$\therefore \overline{100}|^5 y^5 = 2327834559873$$
$$\therefore y^5 = 232{,}7834559873$$
$$\therefore 5 \text{ Log. } y = \text{Log. } 232{,}78345 \text{ nearly}, = 2{,}366952110$$
$$\therefore \text{Log. } y = {,}473390422$$
$$\therefore y = 2{,}9743381 \ldots.$$
$$\therefore x = 297{,}43381 \ldots.$$

The seven first figures are true.

If tables of Logarithms are not at hand, recourse may be had to the mode of discovering the root by gradual approaches to it; for the first figures being found, the unknown number may be made equal to the sum or difference of the known number and another unknown number.

$$\text{Let } x^n = k.$$

And suppose x to lie between the two numbers c and d.

Then suppose $x = c - z$ or $d + z$

$$\therefore \overline{c - z}|^n = k = \overline{d + z}|^n$$

∴

$$\therefore c^n - n c^{n-1} z + n.\frac{n-1}{2}.c^{n-2} z^2 \ldots\ldots = k$$

$$\text{or } d^n + n d^{n-1} z + n.\frac{n-1}{2}\, d^{n-2} z^2 \ldots\ldots = k$$

Neglect all the powers of z.

$$\therefore c^n - n c^{n-1} z = k \quad \therefore z = \frac{c}{n} - \frac{k}{n c^{n-1}} =$$

$$\frac{1}{n}.\overline{\frac{c-k}{c^{n-1}}}$$

$$d^n + n\, d^{n-1} z = k \quad \therefore z = \frac{k}{n d^{n-1}} - \frac{d}{n} =$$

$$\frac{1}{n}.\overline{\frac{k-d}{d^{n-1}}}.$$

Hence, by taking away z thus found from c by the first expression, or by adding it to d as found by the second expression, a number nearer to x in the given equation than either c or d is found. Let these numbers be e or f, and, by proceeding as before, a number still nearer to x is found, and so on for ever.

Let $x^2 = 2$.

If $x = 1$, $x^2 = 1$; and if $x = 2$, $x^2 = 4$. Therefore x is between one and two

Let $x = 1 + z$.

$$\therefore 1 + 2z + z^2 = 2$$

neglecting z^2

$$z = \frac{2-1}{2} = \tfrac{1}{2} = ,5 \text{ which is too great for } z = ,5 - \frac{z^2}{2}$$

$$\therefore x > 1,5.$$

Let $x = 1,5 - z$.

$$\therefore \overline{1,5}|^2 - 2.\ 1,5\, z + z^2 = 2;$$

neglecting

neglecting z^2,

$$\overline{1{,}5}|^2 - 2 = 3{,}0\,z.$$

$$\therefore z = {,}5 \times 1{,}5 - \tfrac{2}{3} = \left\{ \begin{array}{r} {,}75 - \\ - {,}666 \end{array} \right\} = {,}083$$

which is too small.

$$\therefore x > 1{,}5 - {,}083$$

$$\therefore x > 1{,}4166.$$

Since x is less than 1,4166, it may be either greater or less than 1,416. Therefore make $x = 1{,}416 \pm z$;

$$\therefore \overline{1{,}416}|^2 \pm 2 \,.\, 1{,}416 \,.\, z + z^2 = 2\,.$$

$\therefore$ neglecting z^2,

$$\overline{1{,}416}|^2 \backsim 2 = 2 \,.\, 1{,}416\, z$$

$$\therefore z = \frac{1{,}416}{2} \backsim \frac{1}{1{,}416}$$

$$= {,}708 \backsim {,}706214689$$

$$= {,}001785311 \text{ too small}$$

$$\therefore x = 1{,}4142 14688\ldots$$

Make $x = 1{,}414214$, and proceed as before.

Let $x^3 = 37945$.

Suppose $x = 10\,y$.

$$\therefore 10\,y^3 = 37945$$

$$\therefore y^3 = 37{,}945$$

$\therefore y$ is between 3 and 4, but nearest to 3.

Let it therefore be equal to $3 + z$.

$$\therefore 3^3 + 3 \cdot 3^2\, z + 3 \cdot 3\, z^2 + z^3 = 37{,}945,$$

neglecting z^2 and z^3;

$$3 \cdot 3^2\, z = 37{,}945 - 3^3$$

$$\therefore z = \frac{37{,}945}{9 \cdot 3} - 1$$

$$= \frac{4,2161}{3} - 1$$

$$= 1,40537 - 1 = ,40537 \text{ too great.}$$

$$\therefore y > 3,40537.$$

$$\text{Let } y = 3,4 - z.$$

$$\therefore \overline{3,4}|^3 - 3 \cdot \overline{3,4}|^2 z + - - - = 37,945$$

$$\therefore \overline{3,4}|^3 - 37,945 = 3 \cdot \overline{3,4}|^2 \cdot z$$

$$\therefore \frac{3,4}{3} - \frac{37,945}{3 \cdot \overline{3,4}|^2} = z = \begin{array}{r} 1,133333 \\ -\ 1,09414 \end{array}$$

$$= ,039818 \text{ too small.}$$

$$\therefore y > 3,36081.$$

$$\text{Make } y = 3,36 \pm z.$$

From these instances it appears, that by every operation a nearer approach is made to the required root; and in such laborious work it is useful to adopt every method for shortening the number of operations. Now this will be done by a slight attention to the neglected powers of z.

Thus, in the instance $x^2 = 2$, z is found to be $,5 - \frac{z^2}{2}$.

Consequently z cannot be equal to $,5$; nor $\frac{z^2}{2}$ to $\frac{,25}{2}$, or $,125$. Therefore if $,125$ is taken from $,5$ the result $,375$ will give a number less than z. Consequently z is between the numbers $,5$ and $,375$; and it is nearer to $,375$ than to $,5$. Hence, instead of using $1,5 - z$ for the next approach, it would be expedient and justifiable to try $1,4 \pm z$ for x.

∴

$$\therefore \overline{1,4}|^2 \pm 2 \,.\, 1,4 \,.\, z + z^2 = 2$$

$$\therefore 1,4 \pm 2 \,.\, z + \frac{z^2}{1,4} = \frac{2}{1,4} = \frac{1}{,7} = 1,428571.$$

From this equation it appears z is to be added to 1,4; and consequently the equation becomes

$$2\,z = 1,428571 - 1,4 - \frac{z^2}{1,4}$$

$$\therefore z = \frac{,028571428}{2} \ldots - \frac{z^2}{2.\,1,4}$$

$$\therefore z = ,0142857 \ldots - \frac{z^2}{2.\,1,4}.$$

Hence z is less than ,0142857, . . . and z^2 is nearly equal to $\overline{,014}|^2$. Therefore $\frac{z^2}{2.1,4} = \frac{\overline{,014}|^2}{2.1,4}$ nearly $=$ $\frac{,001 \times ,014}{2 \times 0,1} = \frac{,000014}{2.\,0,1} = ,00007$ nearly.

$$\therefore z = \left\{ \begin{array}{l} ,0142857 \\ -,00007 \end{array} \right\} = ,0142157 \text{ too great.}$$

$$\therefore x = 1,414215 \pm z.$$

By proceeding in this manner another approach will be made, which will double the number of figures.

In the instance $y^3 = 37,945$ the equation derived was $3^3 + 3 \cdot 3^2\, z + 3 \cdot 3 z^2 + z^3 = 37,945$ (see page 9)

$$\therefore 3.\,z = \frac{37,945}{9} - 3 - z^2 - \frac{z^3}{9}$$

$$\therefore 3z = \left\{\begin{array}{l} 4{,}2161 \ldots - z^2 - \dfrac{z^3}{9} \\ -3, \end{array}\right\} = 1{,}2161 \ldots\ldots$$

$$- z^2 - \frac{z^3}{9}.$$

$$\therefore z = {,}40537037037 - \frac{z^2}{3} - \frac{z^3}{3\cdot 9}$$

$$\therefore \frac{z^2}{3} = \frac{{,}16}{3} \text{ nearly} = {,}0533$$

$$\frac{z^3}{3\cdot 9} = \frac{{,}064}{3\cdot 9} = {,}00237 \ldots$$

$$\therefore z = \left\{\begin{array}{r} {,}40537037 \\ - \ {,}0557 \end{array}\right\} = {,}3496 \ldots\ldots$$

$$\therefore y = 3{,}35 \pm z$$

$$\therefore \overline{3{,}35}\rceil^3 \pm 3\cdot \overline{3{,}35}\rceil^2 z + 3\cdot 3{,}35\, z^2 \pm z^3 = 37{,}945$$

$$\therefore 3{,}35 \pm 3z + \frac{3\cdot z^2}{3{,}35} \pm \frac{z^3}{\overline{3{,}35}\rceil^2} = \frac{37{,}945}{\overline{3{,}35}\rceil^2} = \frac{7{,}589}{{,}67 \times 3{,}35}$$

$$= \frac{1{,}5178}{{,}67 \times {,}67} = 3{,}381153 \ldots$$

$$\therefore 3z = 0{,}0311538 - \frac{3z^2}{3{,}35} - \frac{z^3}{\overline{3{,}35}\rceil^2}$$

$$\therefore z = 0{,}0103846 - \frac{z^2}{3{,}35} - \frac{z^3}{\overline{3{,}35}\rceil^2}$$

$$\therefore \frac{z^2}{3{,}35} = \frac{\overline{{,}0103}\rceil^2}{3{,}35} = {,}000031669$$

$$\therefore z = \left\{\begin{array}{r} 0{,}0103846 \\ - \ {,}00003166 \end{array}\right\} = {,}01035297 \text{ nearly.}$$

$$\therefore y = \left\{\begin{array}{r} 3{,}35 \\ + \ {,}0103529 \end{array}\right\} = 3{,}3603529 \text{ nearly.}$$

$$\therefore x = 33{,}60352 \pm z.$$

Hence

Hence it appears, that by taking two or three of the figures first found for z, and thence obtaining a number nearly equal to z^2, a correction may be made to z first found; and thus a greater number of true figures will be found in each operation, and consequently the labour upon the whole will be very much shortened.

As the great object in all the operations is to make $\dfrac{\frac{n-1}{2} z^2}{a}$ bear a very small proportion to z; a general expression for the first near number to the root may, when logarithmical tables are not at hand, be applied with great success.

Let $x^n = k$ and a be a number nearly equal to x.

$$\therefore \overline{a+z}|^n = k$$

$$\therefore a^n + n\,a^{n-1} z + n \cdot \frac{n-1}{2}\, a^{n-2} z^2 + \ldots\ldots = k$$

$$\therefore n\,a^{n-1} z + n \cdot \frac{n-1}{2}\, a^{n-2} z^2 + \ldots\ldots = k - a^n$$

$$\therefore z + \frac{n-1}{2} \frac{z^2}{a} + \frac{n-1}{2} \cdot \frac{n-2}{3} \frac{z^3}{a^2} + \ldots\ldots\ldots$$

$$= \frac{k-a^n}{n\,a^{n-1}} = \frac{k-a^n}{n\,a^n} \times a$$

$\therefore$ by neglecting z^2 and the higher powers,

$$z = \frac{k-a^n}{n\,a^n} \times a \text{ nearly.}$$

$\therefore$

$$\therefore a + z = \frac{k - a^n}{n\, a^n} \times a + a.$$

This expression $\frac{k - a^n}{n\, a^n} \times a + a$ is greater than x.

Since $k = a^n + n\, a^{n-1} z + n \cdot \frac{n-1}{2} a^{n-2} z^2 + \ldots$

$\ldots\ldots = \text{A}$.

Multiply both sides by $\overline{n-1}$; and add a^n to each side.

$$\therefore \overline{n-1} \cdot k + a^n = n\, a^n + n \cdot \overline{n-1} \cdot a^{n-1} z + \overline{n-1} \cdot n \cdot \frac{n-1}{2} a^{n-2} z^2 + \ldots = \text{B}.$$

Also multiply the former equation A, both sides by a, and subtract a^{n+1}

$$\therefore \overline{k - a^n} \times a = n\, a^n z + n \cdot \frac{n-1}{2} a^{n-1} z^2 + \ldots = \text{C}$$

$$\therefore \frac{\text{C}}{\text{B}} = \frac{k - a^n}{\overline{n-1} \cdot k + a^n} \times a = \frac{n\, a^n z + n \cdot \frac{n-1}{2} a^{n-1} z^2 + \ldots}{n\, a^n + n \cdot \overline{n-1} \cdot a^{n-1} z \ldots}$$

$$= \frac{n\, a^n z}{n\, a^n} \text{ nearly} = z$$

$$\therefore a + z = a + \frac{k - a^n}{\overline{n-1} \cdot k + a^n} \times a \text{ nearly}.$$

Hence, since $\frac{k - a^n}{n\, a^n} \times a + a$ is greater than x, and $a + \frac{k - a^n}{\overline{n-1} \cdot k + a^n}$ is less than x, by adding together the two upper parts of these fractions for a new upper part, and the

the lower parts for a new lower part, a fraction will be found, which added to a is nearer to x than either of the other two terms*:

$$\overline{k-a^n}\times a+\overline{k-a^n}\times a=\overline{k-a^n}\times 2a=\mathrm{D}$$

$$na^n+\overline{n-1}\,.\,k+a^n=\overline{n+1}\,.\,a^n+\overline{n-1}\,k=\mathrm{E}$$

$$\therefore a+\frac{\mathrm{D}}{\mathrm{E}}=a+\frac{k-a^n}{\overline{n+1}\,.\,a^n+\overline{n-1}\,.\,k}\times 2a.=x\text{ nearly}\dagger.$$

Let

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* If either of two fractions is nearly equal to a given number, the sum of their upper parts, divided by the sum of their lower parts, is still nearer to equality with that number than either of the two fractions.

$$\text{Let } \frac{a}{b}=m+x;$$

$$\frac{c}{d}=m-y;$$

$$\text{Then } a=bm+bx$$

$$\text{And } c=dm-dy$$

$$\therefore a+c=bm+dm+bx-dy$$

$$\therefore \frac{a+c}{b+d}=m+\frac{bx}{b+d}-\frac{dy}{b+d}.$$

But $\frac{bx}{b+d}$ is less than x; and $\frac{dy}{b+d}$ is less than y. Therefore $m+\frac{bx}{b+d}-\frac{dy}{b+d}$ is less than $m+x$; and consequently $\frac{a+c}{b+d}$ is nearer to m than $\frac{a}{b}$ is to m. And $m+\frac{bx}{b+d}-\frac{dy}{b+d}$ is greater than $m-y$. Therefore $\frac{a+c}{b+d}$ is nearer to m than $\frac{c}{d}$ is to m.

† This is De Lagny's expression. Dr. Hutton simplifies this expression in appearance by bringing it to one common lower part:

$a+$

Let $x^2 = 2$.

$$\therefore a = 1;\ k = 2;\ n = 2.$$

$$\therefore a + \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = 1 + \frac{2-1}{\overline{2+1} \cdot \times 1 + 1 \cdot 2}$$

$$\times 2 = 1 + \frac{1}{5} \times 2 = 1 + \frac{2}{5} = 1{,}4.$$

Again let $a = 1{,}4$.

$$\therefore \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = \frac{2 - \overline{1{,}4}|^2}{3 \cdot \overline{1{,}4}|^2 + 2} \times 2 \cdot 1{,}4$$

$$= \frac{2 - \overline{1{,}4}|^2}{3 \cdot {,}7 \times 1{,}4 + 1} \times 1{,}4 = \frac{1 - {,}7 \times 1{,}4}{3 \times {,}7 \times {,}7 + {,}5} \times 1{,}4 =$$

$$\frac{1 - {,}98}{1{,}47 + {,}5} \times 1{,}4 = \frac{{,}02 \times 1{,}4}{1{,}97} = \frac{{,}028}{1{,}97} = {,}0142319.$$

$$\therefore a + \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1}\, k} \times 2a = 1{,}41421319 \ldots$$

$$\ldots\ldots = x.$$

Of these the first seven figures are true; and for the next approach a might be made equal to 1,4142132: but the raising of so great a number to a second power is very laborious; and as the divider also consists of a great number of figures, though the result will give a greater number of figures than by the expression $z = \frac{k}{2a} - \frac{a}{2} - \frac{z^2}{2a}$, yet

---

$$a + \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = \frac{\overline{n+1} \cdot a^{n+1} + \overline{n-1} \cdot ak + 2ak - 2a^{n+1}}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k}$$

$= \frac{\overline{n+1} \cdot k + \overline{n-1} \cdot a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times a$; but he increases the trouble of the operation.

this

this latter expreſſion will be found far more convenient in practice.

$$\text{Let } x^3 = 37{,}945.$$

$$\therefore\ a = 3\ k = 37{,}945\ n = 3$$

$$\therefore \frac{\overline{k - a^3} \times 2{,}a}{\overline{n+1}\,.\,a^n + \overline{n-1}\,.\,k} = \frac{37{,}945 - 27}{4 \cdot 27 + 2 \cdot 37{,}945} \times 6 =$$

$$\frac{10{,}945 \times 3}{2 \times 27 + 37{,}945} = \frac{2{,}189 \times 3}{,4 \times 27 + 7{,}589} = \frac{2{,}189 \times 3}{18{,}389}$$

$$= \frac{,3127\overline{142857}1 \times 3}{2{,}627} = ,119038 \times 3 = ,3571$$

$$\therefore\ x = 3{,}3571.$$

Now let $a = 3{,}35$,

and the learner by purſuing a ſimilar operation may compare the effects of it with De Lagny's method.

The mode of dividers may alſo be applied to equations in this claſs, for the diſcovery of a few of the firſt figures. Thus, let $x^2 = 2$; therefore if

x	$=$	1	2	1,5	1,4	1,414	1,415
$\frac{2}{x}$	$=$	2	1	1,33	1,428	1,414422	1,413427
Diff.	$=$			,16	,028	,000422	,001572.

Hence, x is between 1,5 and 1,40 but nearer to 1,4, and the diff. ,16 + ,028 = ,188.*

∴

* The ſum of the differences in the third line may be divided into ten, twenty, thirty, &c. equal parts, according as their correſponding numbers in the upper line differ by ,1 ,2 ,3, &c. and as often as one of theſe parts is contained in either of the lower differences, ſo many parts of ,1

$$\therefore \frac{\text{Diff.}}{10} = ,0188$$

$$\text{and } 8 \times ,0188 = ,151 > ,16$$

$$\therefore 1,5 - ,08 \ldots = x = 1,41 \ldots.$$

$$\text{and } ,0188 \times 1 > ,028$$

$$\therefore 1,4 + ,01 \ldots. = x = 1,41 \ldots.$$

$$\text{Alſo } \frac{,16}{,0188} = 8,51$$

$$\therefore 1,5 - ,0851 . = 1,4148 \ldots = x \text{ nearly.}$$

$$\frac{,028}{,0188} = 1,489$$

$$\therefore 1,4 + ,01489 = 1,414 \ldots = x \text{ nearly.}$$

,2 ,3, &c. may be added to or taken from the correſponding upper numbers, and the reſult will be a number much nearer to the root required. Or, ſince the root lies between two numbers in the upper line, and the differences in the lower line are very nearly in the proportion of the exceſs of one number in the upper line above the root to the defect of the other number from the root, the following proportion will be nearly true:

$$,16 : ,028 :: ,1 - z : z$$

$$\therefore ,16\, z = ,0028 - ,028\, z$$

$$\therefore ,188\, z = ,0028$$

$$\therefore z = \frac{,0028}{,188} = ,01489$$

$$\therefore x = 14 + ,01489 = 14,1489 \text{ nearly.}$$

Again:

$$,028 : ,16 :: ,1 - z : z$$

$$\therefore ,028\, z = ,016 - ,16\, z$$

$$\therefore ,188\, z = ,016$$

$$z = \frac{,016}{,188} = ,0851$$

$$\therefore x = 1,5 - ,c851 = 1,4148 \ldots \text{ nearly.}$$

For

For the first approach therefore x should be made $1,414 + z$

$$\text{Let } x^3 = 37,945.$$

$$\begin{array}{rcllll} x & = & 3 & 4 & 3,3 \\ \frac{37,945}{x} & = & 12,648 . & 9,4 & 11,49 \\ x^2 & = & 9 & 16 & 10,89 \\ \text{Diff.} & = & 3,648 & & 0,60 \end{array}$$

$$\frac{3,648 + 0,60}{30} = \frac{4,248}{30} = 1,416$$

$$,4 \times 1,416 = ,5764 > ,60$$

$$\therefore x = 3,3 + ,04 \ldots\ldots = 3,34 \ldots\ldots$$

For the first approach therefore make $x = 3,34 + z$.

$$\text{Let } y^5 = 232,78345559873 = k.$$

$$\begin{array}{rcll} y & = 2 & 3 \\ \frac{k}{y} & = 116 & 77 \\ y^4 & = 16 & 81. \end{array}$$

Here y is very nearly equal to 3, and it will be better to make $y = 3 - z$ than to attempt further approaches by dividers, as the next trial must be made with 2,9 or 2,96.

CHAP. III.

EQUATIONS OF THE SECOND CLASS.

Equations of the ſecond claſs have two unknown terms, and are all reducible to one or other of theſe forms.

$$1.\ x^{m+n} + a\,x^n = k$$
$$2.\ x^{m+n} - a\,x^n = k$$
$$3.\ a\,x^n - x^{m+n} = k$$

The two firſt forms admit only of one root; the third form admits of two roots. The firſt form admits only of one root: for $x^{m+n} + a\,x^n = x^n \times \overline{x^m + a}$, and both parts x^n, and $x^m + a$ increaſe or decreaſe together; and conſequently the product increaſes or decreaſes with the increaſe or decreaſe of x: and, if this product is in any inſtance equal to k, by increaſing or decreaſing x the product muſt be made greater or leſs than k.

In the ſame manner $x^{m+n} - a\,x^n$ in the ſecond form is equal to $x^n \times \overline{x^m - a}$, whoſe parts alſo both increaſe or decreaſe together, and conſequently there can be but one root to an equation in this form.

But $a\,x^n - x^{m+n}$ in the third form is equal to $x^n \times \overline{a - x^m}$, whoſe parts do not increaſe and decreaſe together: for, if x^n increaſes, $a - x^m$ decreaſes; and, if x^n decreaſes, $a - x^m$ increaſes. Conſequently, if this product is equal to k, when $x = d$, by increaſing x the product may firſt increaſe and then decreaſe, and conſequently

quently another number c greater than d may make $x^n \times \overline{a - x}^m = k$. But this form does not admit of more than two roots. For the product increaſes by addition to x, from the leaſt aſſignable number till it becomes the greateſt poſſible, and then by farther addition to x it decreaſes, till the product becomes nothing, and *vice verſa*.

To prove this, let $\dot{x}$ be a very ſmall number, ſo ſmall that its powers multiplied into any given number ſhall be leſs than any aſſignable number, and let $\dot{x}$ added to or taken from $x = y$.

To take the ſimpleſt caſe, let $a x - x^2 = k$ and $y = x + \dot{x}$.

$$\therefore a y = a x + a \dot{x}$$
$$y^2 = x^2 + 2 x \dot{x} + \dot{x}^2$$
$$\therefore a y - y^2 = a x - x^2 + a \dot{x} - 2 x \dot{x} \text{ nearly}$$
$$\therefore \overline{a y - y^2} \backsim \overline{a x - x^2} = a \dot{x} \backsim 2 x \dot{x} \text{ nearly.}$$
$$\text{If } y = x - \dot{x}$$
$$a y = a x - a \dot{x}$$
$$y^2 = x^2 - 2 x \dot{x} + \dot{x}^2$$
$$\therefore a y - y^2 = a x - x^2 - a \dot{x} + 2 x \dot{x} \text{ nearly}$$
$$\therefore \overline{a y - y^2} \backsim \overline{a x - x^2} = 2 x \dot{x} \backsim a \dot{x} \text{ nearly.}$$

In the firſt caſe, by adding $\dot{x}$ the term $a y - y^2$ is greater than $a x - x^2$, if a is greater than $2 x$ or $\frac{a}{2}$ greater than x: but, if a is leſs than $2 x$, or $\frac{a}{2}$ leſs than x, then the term $a y - y^2$ is leſs than $a x - x^2$. Hence, the term

term $a\,x - x^2$ is increaſed by adding to x as long as x is leſs than $\frac{a}{2}$; but it is decreaſed by adding to x as long as x is greater than $\frac{a}{2}$.

In the ſecond caſe, by taking away $\dot{x}$ the term $a\,y - y^2$ is greater than $a\,x - x^2$, when $2\,x$ is greater than a, or x is greater than $\frac{a}{2}$: but, if $2\,x$ is leſs than a, or x leſs than $\frac{a}{2}$, then $a\,y - y^2$ is leſs than $a\,x - x^2$. Hence, the term $a\,x - x^2$ is decreaſed by taking from x as long as x is leſs than $\frac{a}{2}$; but it is increaſed by taking from x as long as x is greater than $\frac{a}{2}$.

If therefore in the equation $a\,x - x^2 = k$, d leſs than $\frac{a}{2}$ is one root, then no other number leſs than $\frac{a}{2}$ can be a root; for, by adding to or taking away from d numbers till $d \pm z$ is equal to $\frac{a}{2}$, the term $a\,x - x^2$ muſt equal a number greater or leſs than k. But ſince, by adding to d till x becomes equal to $\frac{a}{2}$, the term $a\,x - x^2$ is increaſed, and then by further addition is decreaſed till it becomes nothing, there muſt be another number which will make $a\,x - x^2$ equal to k: and there can be only one

one number greater than $\frac{a}{2}$; for by adding now to x, the term $a x - x^2$ is constantly diminished.

The same may be proved generally.

Thus let $a x^n - x^{m+n} = k$.

$$\therefore x^n \times \overline{a - x^m} = k.$$

Let $y = x \pm \dot{x}$.

$\therefore y^n \times \overline{a - y^m} \backsim x^n \times \overline{a - x^m}$ is equal to the increase or decrease of $x^n \times \overline{a - x^m}$ by substituting a number for x but nearly equal to it in the equation $a x^n - x^{m+n} = k$.

$$y^n = x^n \pm n x^{n-1} \dot{x} \text{ nearly;}$$
$$a - y^m = a - x^m \mp m x^{m-1} \dot{x} \text{ nearly;}$$
$$\begin{aligned} \therefore y^n \times \overline{a - y^m} = a x^n &\pm n a x^{n-1} \dot{x} \\ - x^{m+n} &\mp n x^{m+n-1} \dot{x} \\ &\mp m x^{m+n-1} \dot{x} \end{aligned}$$

$\therefore y^n \times \overline{a - y^m} \backsim x^n \times \overline{a - x^m} = n a x^{n-1} \dot{x} \backsim n x^{m+n-1} \dot{x} + m x^{m+n-1} \dot{x} = x^{n-1} \dot{x} \times n a \backsim \overline{n + m} \cdot x^m$.

Hence, the term $x^n \times \overline{a - x^m}$ constantly increases or decreases, whilst $n a$ is greater or less than $\overline{n + m} \cdot x^m$; but if it increases when $n a$ is greater than $\overline{n + m} \cdot x^m$, it decreases when $\overline{n + m} \cdot x^m$ is greater than $n a$: and, if it decreases when $n a$ is less than $\overline{n + m} \cdot x^m$, it increases when $\overline{n + m} \cdot x^m$ is less than $n a$.

Instances.

Inftances.

$$8x - x^2 = 15.$$

The roots are 5 and 3.

Subftitute 1, 2, 4, 6, 7, for x

$$\therefore 8x - x^2 = 2 \times \overline{8-2} = 12$$
$$8x - x^2 = 1 \times \overline{8-1} = 7$$
$$8x - x^2 = 4 \times \overline{8-4} = 16$$
$$8x - x^2 = 6 \times \overline{8-6} = 12$$
$$8x - x^2 = 7 \times \overline{8-7} = 7.$$

In this cafe $na = 8$ and $\overline{n+m} \,.\, x^n = 2x$.

Therefore, when $n\ a$ is greater than $\overline{n+m} \,.\, x^m$, 8 is greater than $2x$, or 4 is greater than x; and the numbers 1, 2, 3 fubftituted for x give refults lefs than 4, but increafing; 5, 6, 7 give alfo refults lefs than 4, and decreafing.

$$\text{Let } \frac{61}{9} x^2 - x^3 = \frac{400}{9},$$

whofe roots are 4 and 5;

$$\text{let } x = 1, \text{ then } \frac{61}{9} x^2 - x^3 = \frac{52}{9}$$

$$x = 2 \qquad = \frac{172}{9}$$

$$x = 3 \qquad = \frac{306}{9}$$

$$x = 6 \qquad = \frac{252}{9}$$

$x = 7$ Impoffible.

The

The numbers 1, 2, 3, 4, give results increasing ; a number between 4 and 5 makes the result decrease : 5, 6 continue to make decreasing results ; if 7 is tried, there is no result.

Let $n = 1$ and $m = 2$.

$$\therefore a x^n - x^{m+n} = a x - x^2 = k.$$

Since this form admits of two roots, let them be c and d

$$\therefore a c - c^2 = k = a d - d^2$$
$$\therefore a c - a d = c^2 - d^2$$
$$\therefore a \times \overline{c - d} = c^2 - d^2$$
$$\therefore a = \frac{c^2 - d^2}{c - d} = c + d.$$

Hence, in an equation of this form a the copart of x is always equal to the sum of the roots.

Since also $a = c + d$

$$a c = c^2 + c d$$
$$\therefore a c - c^2 = c d$$
but $a c - c^2 = k$
$$\therefore k = c d.$$

Hence k the known term is in this form always equal to the product of the roots. Thus let $12 x - x^2 = 32$.

The roots are 8 and 4, and
$8 + 4 = 12 = a$, and $32 = 8 \times 4 = k$.

But in certain cafes equations in this form will only have one root; namely, when the root is one half of a. For let m be the number which makes $a\,x - x^2 = G$, or the greateft poffible number. Therefore, if x is made equal to $m + z$ or $m - y$, the product refulting will be lefs, and in both cafes may be equal to a number k.

$$\therefore \begin{matrix} a\,x \\ - x^2 \end{matrix} = \left\{ \begin{matrix} a\,m + a\,z \\ - m^2 - 2\,m\,z - z^2 \end{matrix} \right\} = \begin{matrix} a\,m - a\,y \\ - m^2 + 2\,m\,y - y^2 \end{matrix} \Bigg\} = k$$

$$\therefore a\,z + a\,y = 2\,m\,z + 2\,m\,y + z^2 - y^2$$

$$\therefore a \times \overline{z + y} = 2\,m \times \overline{z + y} + z^2 - y^2$$

$$\therefore a = 2\,m + z - y.$$

Now, fince m and a are conftant; but z or y may be taken at pleafure, $z - y$ muft be either an invariable number, or equal to nothing.

But $z - y$ cannot be an invariable number, for, by decreafing z and y, the number $z - y$ may be lefs than any affignable number. Therefore z muft be equal to y, and confequently $a = 2\,m$.

$$\therefore m = \frac{a}{2}.$$

Confequently, if the root of the equation $a\,x - x^2 = k$ is $\frac{a}{2}$ there can be only one root. In this cafe $a \times \frac{a}{2} - \frac{a^2}{4} = k = \frac{a^2}{4}$.

Therefore, if an equation of this form has only one root, that root is one half of the copart of x, and the known

known term k is the second power of half the co-part a.

If $n = 1$ and $m = 3$; then

$$a x^n - x^m = k = a x - x^3.$$

Let m be the number which, placed for x, makes $a x - x^3 = G$ the greatest possible. Therefore, as before, $m + z$ or $m - y$, substituted for x, will make $a x - x^3 = k$.

$$\therefore \left.\begin{aligned} a x &= a m + a z \\ - x^3 &= - m^3 - 3 m^2 z - 3 m z^2 - z^3 \end{aligned}\right\} =$$

$$\left.\begin{aligned} &a m - a y \\ &- m^3 + 3 m^2 y - 3 m y^2 + y^3 \end{aligned}\right\} = k$$

$$\therefore a z + a y = 3 m^2 y + 3 m^2 z + 3 m z^2 - 3 m y^2 + z^3 + y^3$$

$$\therefore a \times \overline{z + y} = 3 m^2 \times \overline{y + z} + 3 m \times \overline{z^2 - y^2} + z^3 + y^3.$$

$$\therefore a = 3 m^2 + 3 m \times \overline{z - y} + \frac{z^3 + y^3}{z + y}.$$

But $3 m \times \overline{z - y} + \frac{z^3 + y^3}{z + y}$ must be equal to nothing.

$$\therefore a = 3 m^2$$

$$\therefore m = \frac{\sqrt{a}}{3}.$$

Example.

$$27 x - x^3 = 54.$$

In this case the equation can have only one root; for $x = 3 = \sqrt{\frac{27}{3}}$.

If

If x is made equal to one;

Then $27\,x - x^3 = 26$; and this equation will have another root $= 4{,}6235$ nearly.

$$\therefore y = 2 \text{ and } z = 1{,}6235$$
$$\text{but } y^3 + z^3 = 3\,m \times \overline{y^2 - z^2}$$
$$\therefore 8 + \overline{1{,}6235}|^3 = 9 \times \overline{4 - \overline{1{,}6235}|^2}$$
$$\therefore \overline{1{,}6235}|^2 \times \overline{9 + 1{,}6235} = 36 - 8 = 28$$
$$\therefore \overline{1{,}6235}|^2 \times 10{,}6235 = 28.$$

Take the three first figures of each part of the compound term 1,62 and 1,06.

1,62	2,6244
1,62	1,06
324	157464
972	262440
162	2,781864
2,6244	

Had more figures been taken for z, the product would have approached nearer to 28.

In the same manner, if $n = 1$ and $m = 3$ that is if $a\,x - x^4 = k$, it may be discovered whether the equation has one or two roots. For if it has only one root, that root will be equal to $\sqrt[3]{\frac{a}{4}}$.

And in general it may be discovered whether an equation of the second class has two roots or not. Thus

$$a\,x^n - x^{m+n} = k.$$

Let

Let M be the number which makes $a\,x_n - x^{m+n} = G$ or the greatest possible number.

1st. Let $x = M + y$.

$$\therefore \begin{matrix} a\,x^n \\ - x^{m+n} \end{matrix} \left\{ \begin{matrix} a\,M^n + n\,a\,M^{n-1}\,y + n\,.\,\frac{n-1}{2}a\,M^{n-2}\,y^2 \\ - M^{m+n} - \overline{m+n}.M^{m+n-1}y - \overline{m+n}.\frac{m+n-1}{2} \end{matrix} \right.$$

$$\left. \begin{matrix} + \ldots\ldots\ldots \\ M^{m+n-2}\,y^2 + . \end{matrix} \right\} k$$

2d. Let $x = M - z$.

$$\therefore \begin{matrix} a\,x^n \\ - x^{m+n} \end{matrix} \left\{ \begin{matrix} a\,M^n + n\,a\,M^{n-1}\,z + n\,.\,\frac{n-1}{2}\,a\,M^{n-2} \\ - M^{m+n} + \overline{m+n}\,.\,M^{m+n-1}\,z - \overline{m+n}\,. \end{matrix} \right.$$

$$\left. \begin{matrix} z^2 + \ldots\ldots \\ \frac{m+n-1}{2}\,M^{m+n-2}\,z^2 + \ldots \end{matrix} \right\} k$$

$\therefore n\,a\,M^{n-1} \times \overline{z+y} = \overline{m+n}\,.\,M^{m+n-1} \times \overline{z+y} + A\,z^2 - B\,z^3 + C\,.\,z^4.$

$$\text{But } A\,z^2 - Bz^3 + C\,z^4 \ldots\ldots = 0$$

$$\therefore n\,a\,M^{n-1} \times \overline{z+y} = \overline{m+n}\,.\,M^{m+n-1} \times \overline{z+y}$$

$$\therefore n\,a\,M^{n-1} = \overline{m+n}\,.\,M^{m+n-1}$$

$$\therefore \frac{n\,a}{m+n} = M^m$$

$$\therefore M = \sqrt[m]{\frac{n\,a}{m+n}}.$$

Since

ince $M = \sqrt[m]{\frac{n\,a}{m+n}}$.

$$M^m = \frac{n\,a}{m+n}$$

$$\therefore a = M^m \times \frac{m+n}{n}.$$

But, ſince $a\,x^n - x^{m+n} = k$

$$a = \frac{k + x^{m+n}}{x^n}$$

That is, when $x = M$

$$a = \frac{k + M^{m+n}}{M^n} = M^m \times \frac{m+n}{n}$$

$$\therefore k + M^{m+n} = M^{m+n} \times \frac{m+n}{n}$$

$$\therefore k = M^{m+n} \times \frac{m+n}{n} - M^{m+n} = M^{m+n} \times \frac{m}{n}$$

To find the relation between a, k, and the roots of the equation $a\,x^n - x^{m+n} = k$.

Let the roots be c and d.

$$\therefore a\,c^n - c^{m+n} = k = a\,d^n - d^{m+n}$$

$$\therefore a\,c^n - a\,d^n = c^{m+n} - d^{m+n}$$

$$\therefore a \times \overline{c^n - d^n} = c^{m+n} - d^{m+n}$$

$$\therefore a = \frac{c^{m+n} - d^{m+n}}{c^n - d^n}$$

$$\therefore a = c^m + d^m + \frac{c^m\,d^n - d^m\,c^n}{c^n - d^n}.$$

Hence,

Hence, if m is equal to n, a is equal to the ſum of the m^{th} powers of the roots of the equation; for in this caſe $c^m d^n - d^m c^n = 0$.

If m is greater than n, a is greater than the ſum of the m^{th} powers of the roots. For in this caſe $\frac{c^m d^n - d^m c^n}{c^n - d^n}$ is a number to be added to $c^m + d^m$.

If m is leſs than n, then $d^m c^n$ is greater than $c^m d^n$, and a number equal to $\frac{d^m c^n - c^m d^n}{c^n - d^n}$ is to be taken away from $c^m + d^m$.

Inſtances.

1ſt. $m = n$.

$$12x - x^2 = 32.$$

The roots 8 and 4

$$a = c + d = 8 + 4 = 12.$$

2d. $m < n$

$$27x - x^3 = 26.$$

The roots are 1 and 4,6235 . . . .

$$a < 1 + 4{,}6235 \text{ or } 5{,}6235$$

3d. $m > n$

$$\frac{61}{9} x^2 - x^3 = \frac{400}{9}.$$

The roots are 4 and 5.

$$4 + 5 = 9 = \frac{81}{9} < \frac{61}{9} \text{ or } a.$$

Since

Since $a x^n - x^{m+n} = k$

$$a = \frac{k + x^{m+n}}{x^n} = \frac{k + c^{m+n}}{c^n} = \frac{k + d^{m+n}}{d^n}.$$

$$\therefore\ k d^n + c^{m+n} d^n = k c^n + d^{m+n} c^n$$

$$\therefore\ c^{m+n} d^n - d^{m+n} c^n = k c^n - k d^n$$

$$\therefore\ c^n d^n \times \overline{c^m - d^m} = k \times c^n - d^n$$

$$\therefore\ k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n}.$$

Hence, if $m = n$, k is equal to the product of the nth powers of the roots.

If m is greater than n, k is greater than the product of the n^{th} powers of the roots.

If m is lefs than n, k is lefs than the product of the n^{th} powers of the roots.

Inftances.

1ft, $12\,x - x^2 = 32$

The roots are 8 and 4.

$k = 32 = c\,d = 8 \times 4.$

Let $27\,x - x^3 = 26.$

The roots are 1 and 4,6235 and m is greater than n.

$k = 26 < 1 \times 4{,}6235.$

Let $\frac{61}{9} x^2 - x^3 = \frac{400}{9}.$

The

The roots are 4 and 5, and m is less than n; $k = \frac{400}{9} > 4^2 \times 5^2$ or 16×25.

To make equations of the form $a\,x^n - a^{m+n} = b$ with either one or two roots.

Make $a = c^m + d^m \pm c^n\,d^n \times \frac{c^{m-n} - d^{m-n}}{c^n - d^n}$.

and make $k = c^n\,d^n \times \frac{c^m - d^m}{c^n - d^n}$.

Instances.

Make an equation of the form $a\,x - x^2 = k$ with the two roots 7 and 9.

$$a = 7 + 9 = 16$$
$$k = 7 \times 9 = 63$$
$$\therefore\ 16\,x - x^2 = 63.$$

Make an equation of the form $a\,x - x^3 = k$ with the roots 7 and 2.

$$n = 1;\ m = 2.$$

$a = c^m + d^m \frac{+ c^m\,d^n - d^m\,c^n}{c^n - d^n} = c^2 + d^2 + \frac{c^2\,d - d^2\,c}{c - d} = c^2 + d^2 + c\,d \times \frac{c - d}{c - d} = c^2 + c\,d + d^2 = 49 + 14 + 4 = 67$

$$k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c\,d \times \frac{c^2 - d^2}{c - d} = c\,d \times \overline{c + d}$$

$$= 14 \times 9 = 126$$

$$\therefore\ 67\,x - x^3 = 126.$$

Make an equation of the form $a\,x^2 - x^3 = k$ with the roots 5 and 2.

$$n = 2;\ m = 1.$$

$$a = c^m + d^m + \frac{c^m\,d^n - d^m\,c^n}{c^n - d^n} = c + d + \frac{c\,d^2 - d\,c^2}{c^2 - d^2}$$

$$= c + d - c\,d \times \frac{c - d}{c^2 - d^2} = c + d - \frac{c\,d}{c + d} =$$

$$5 + 2 - \frac{10}{7} = 7 - 1\tfrac{3}{7} = 5\tfrac{4}{7}$$

$$k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c^2 d^2 \times \frac{c - d}{c^2 - d^2} = \frac{c^2\,d^2}{c + d} =$$

$$\frac{25 \times 4}{7} = \frac{100}{7} = 14\tfrac{2}{7}$$

$$\therefore\ 5\tfrac{4}{7}\,x^2 - x^3 = 14\tfrac{2}{7}.$$

Make an equation of the form $a\,x^3 - x^5 = k$ with the roots 7 and 6.

$$n = 3;\ m = 2.$$

$$a = e^m + d^m + \frac{c^m\,d^n - c^n\,d^m}{c^n - d^n} = c^2 + d^2 + \frac{c^2\,d^3 - c^3\,d^2}{c^3 - d^3}$$

$$= c^2 + d^2 - c^2\,d^2 \times \frac{c - d}{c^3 - d^3} = c^2 + d^2 - \frac{c^2\ d^2}{c^2 + c\,d + d^2}$$

$$= 49 + 36 - \frac{49 \times 36}{49 + 42 + 36} = 85 - \frac{1764}{127} =$$

$$85 - 13\tfrac{113}{127} = 71\tfrac{14}{127}$$

$$k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c^3 d^3 \times \frac{c^2 - d^2}{c^3 - d^3} = c^3 d^3 \times$$

$$\frac{c + d}{c^2 + c d + d^2} =$$

$$7^3 \times 6^3 \times \frac{13}{49 + 42 + 36} = \frac{7^3 \times 6^3 \times 13}{127} = 7583 \tfrac{103}{127}.$$

$$\therefore 71 \tfrac{14}{127} x^3 - x^5 = 7583 \tfrac{103}{127}.$$

Make an equation of the form $a x^2 - x^5 = k$, having only one root 3.

$$n = 2\,;\ m = 3.$$

$$M = \sqrt[m]{\frac{n a}{m + n}} = \sqrt[3]{\frac{2 a}{5}} = 3.$$

$$\therefore \frac{2 a}{5} = 3^3 = 27$$

$$\therefore a = \frac{135}{2} = 67{,}5$$

$$k = M^{m+n} \times \frac{m}{n}$$

$$\therefore k = 3^5 \times \frac{3}{2} = \frac{729}{2} = 364{,}5$$

$$\therefore 67{,}5\, x^2 - x^5 = 364{,}5.$$

The ſame might have been done from the general expreſſions for a and k in terms of c and d, by ſuppoſing that, when there is only one root, the two terms c and d approach nearer to equality than by any given difference. Thus

$$a = c^m + d^m + \frac{c^m d^n - c^n d^m}{c^n - d^n} = c^3 + d^3 + \frac{c^3 d^2 - c^2 d^3}{c^2 - d^2}$$

$$= c^3 + d^3 + c^2 d^2 \times \frac{c - d}{c^2 - d^2} = c^3 + d^3 + \frac{c^2 d^2}{c + d}$$

If d is ſuppoſed to be nearly equal to c;

$$a = 2c^3 + \frac{c^4}{2c} = 2c^3 + \frac{c^3}{2} = \frac{5c^3}{2} = \frac{5 \cdot 27}{2} = 67,5.$$

Alſo $k = c^n d^n \times \dfrac{c^m - d^m}{c^n - d^n} = c^2 d^2 \times \dfrac{c^3 - d^3}{c^2 - d^2} =$ $c^2 d^2 \times \dfrac{c^2 + cd + d^2}{c + d}$ and d being ſuppoſed nearly equal to c

$$k = c^4 \times \frac{3c^2}{2c} = c^4 \times \frac{3c}{2} = \frac{3 \cdot c^5}{2} = \frac{3 \cdot 3^5}{2} = 364,5.$$

In the ſame manner that the relation between a, k, and the roots of the equation $ax^n - x^{m+n} = k$ has been diſcovered; the relation between a, k, and the roots of the equations $x^{m+n} + ax^n = k$, and $x^{m+n} - ax^n = k$ may be inveſtigated. Thus let d be the root of the equation $x^{m+n} + ax^n = k$, and c the root of the equation $x^{m+n} - ax^n = k$; then

$$c^{m+n} - ac^n = k = d^{m+n} + ad^n.$$

$$\therefore c^{m+n} - d^{m+n} = ac^n + ad^n = a \times \overline{c^n + d^n}$$

$$\therefore a = \frac{c^{m+n} - d^{m+n}}{c^n + d^n} = c^m - \frac{c^m d^n}{c^n + d^n} - d^m + \frac{c^n d^m}{c^n + d^n}$$

$$= c^m - d^m - c^n d^n \times \frac{c^{m-n} - d^{m-n}}{c^n + d^n}.$$

Alſo

Alſo, ſince $d^{m+n} + a\,d^n = k$, $a = \dfrac{k - d^{m+n}}{d^n}$; and ſince $c^{m+n} - a\,c^n = k$, $a = \dfrac{c^{m+n} - k}{c^n}$.

$$\therefore \frac{k - d^{m+n}}{d^n} = \frac{c^{m+n} - k}{c^n}$$

$$\therefore k\,c^n - d^{m+n}\,c^n = c^{m+n}\,d^n - k\,d^n$$

$$\therefore k \times \overline{c^n + d^n} = c^{m+n}\,d^n + d^{m+n}\,c^n = c^n\,d^n \times \overline{c^m + d^m}$$

$$\therefore k = c^n\,d^n \times \frac{c^m + d^m}{c^n + d^n}.$$

Hence, if $m = n = 1$, $a = c - d$ and $k = c\,d$.

Let $x^2 + 3x = 40 = y^2 - 3y$.

$$\begin{aligned} &\text{Here } c = 8 \text{ and } d = 5 \\ &a = 3 \;\; = 8 - 5 \\ &k = 40 = 8 \times 5. \end{aligned}$$

A relation alſo is to be found between the roots of the equation $x^{m+n} - a\,x^n = k$ and $a\,y^n - y^{m+n} = k$, which may ſometimes be of uſe.

$$x^{m+n} - a\,x^n = a\,y^n - y^{m+n}$$

$$\therefore x^{m+n} + y^{m+n} = a\,y^n + a\,x^n = a \times \overline{y^n + x^n}$$

$$\therefore a = \frac{x^{m+n} + y^{m+n}}{x^n + y^n}.$$

In certain caſes $x^{m+n} + y^{m+n}$ is diviſible by $x^n + y^n$; in which caſes the relation between x and the roots c and d of the equation $a\,y^n - y^{m+n}$ is eaſily diſcovered. Thus let

let $m = 2$, and $n = 1$.

$$\therefore a = \frac{x^3 + y^3}{x + y} = x^2 - yx + y^2 = x^2 - cx + c^2 =$$

$$x^2 - dx + d^2$$

$$\therefore c^2 - d^2 = cx - dx = x \times \overline{c - d}$$

$$\therefore x = c + d.$$

Hence, the root of the equation $x^3 - ax = k$ is equal to the ſum of the roots of the equation $ax - x^3 = k$.

Alſo a relation may be found between k and the roots of theſe two equations. For

$$a x^n = x^{m+n} - k \quad \therefore a = \frac{x^{m+n} - k}{x^n}$$

$$\text{and } a y^n = y^{m+n} - k \quad \therefore a = \frac{y^{m+n} - k}{y^n}$$

$$\because \frac{x^{m+n} - k}{x^n} = \frac{c^{m+n} - k}{c^n} = \frac{d^{m+n} - k}{d^n}$$

$$\therefore x^{m+n} c^n - k c^n = x^n c^{m+n} - k x^n$$

$$\text{and } x^{m+n} d^n - k d^n = x^n d^{m+n} - k x_n$$

$$\therefore x^{m+n} \times \overline{c^n - d^n} - k \times \overline{c^n - d^n} = x^n \times \overline{c^{m+n} - d^{m+n}}$$

$$\therefore x^{m+n} - x^n \times \frac{c^{m+n} - d^{m+n}}{c^n - d^n} = k.$$

Let $n = 1$ and $m = 2$.

$$\therefore k = x^3 - x \times \frac{c^3 - d^3}{c - d} = x^3 - x \times \overline{c^2 + cd + d^2}$$

$$= x^3 - x \times \overline{c \times \overline{c + d} + d^2} = x^3 - c x^2 - d^2 x = x^2$$

$\times$

$\times \overline{x - c} - d^2 x = x^2 d - d^2 x = d x \times \overline{x - d} = c d x.$

Hence, the root of an equation $x^3 - a x = k$ is equal to the ſum of the roots of the equation $a x - x^3 = k$; and k is equal to the product of the roots of the two equations.

Inſtance.

$$x^3 - 76 x = 240.$$

The root of this equation is 10; and the roots of the equation $76 x - x^3 = 240$ are ſix and four, and $240 = 6 \times 4 \times \overline{6 + 4}$.

A ſimilar relation exiſts between the roots of the equations $x^{m+n} + a x^n = k$, $a y^n - y^{m+n} = k$ and a and k.

For $x^{m+n} + y^{m+n} = a y^n - a x^n = a \times \overline{y^n - x^n}$

$$\therefore a = \frac{x^{m+n} + y^{m+n}}{y^n - x^n}.$$

$$\text{Alſo } a = \frac{k - x^{m+n}}{x^n} = \frac{k + y^{m+n}}{y^n}$$

$$\therefore k y^n - x^{m+n} y^n = k x^n + y^{m+n} x^n$$

$$\therefore k \times \overline{y^n - x^n} = y^{m+n} x^n + x^{m+n} y^n$$

$$\therefore k = \frac{y^{m+n} x^n + x^{m+n} y^n}{y^n - x^n}.$$

To

To find the root or roots of the equation $a\,x^{n} - x^{m+n} = k$.

Firſt try, whether the equation has one or two roots by the expreſſions given for a and k. If it has only one root, that root is found from either of thoſe expreſſions. If it has two roots; try firſt, whether the leaſt root is a whole number, by applying numbers leſs than M already found: and having thus found the leaſt root, the other root may in ſome caſes be eaſily found by the expreſſions for a and k. If the leaſt root is not a whole number, take the difference between M and the neareſt whole number to it, which difference being added to M gives a number very near the greateſt root. By trial it will then be found, whether the greateſt root is a whole number; and, if it is, the leaſt root may alſo, in ſome caſes, be eaſily found from the expreſſions for a and k. If neither of the roots are whole numbers, they will be found beſt by the method of approach.

Let $39\,y - y^{3} = 70$.

$$\therefore \mathrm{M} = \sqrt{\frac{39}{3}} = \sqrt{13} = 3{,}6\ \ldots .$$

$$\mathrm{M} = \sqrt[3]{\frac{k}{2}}$$

$$\text{but } \sqrt[3]{\frac{70}{2}} = \sqrt[3]{35} = 3{,}2.$$

Hence, the equation has two roots.

$$\begin{array}{lllll} y & = 3 & 2 & 5 \\ \dfrac{70}{3} & = 23\tfrac{1}{3}, & 35 & 14 \\ 39 - y^{2} & = 30 & 35 & 14. \end{array}$$

By trying 3 according to the mode of dividers, it appears, that 3 is too great for the leaſt root: and on trying 2, it ſucceeds; conſequently 2 is one root, and as the greateſt root is not ſo far from M as the leaſt is, the greateſt root muſt be very near to if not five. Conſequently 5 was tried for the next ſtep, and the two roots are found, namely 2 and 5.

Let $25\,y - y^3 = 36$.

$$M = \sqrt{\frac{25}{3}} = \sqrt{8,3}$$

$$M = \sqrt[3]{\frac{k}{2}} = \sqrt[3]{\frac{36}{2}} = \sqrt[3]{18}.$$

But $\sqrt{8,3}$ is not equal to $\sqrt[3]{18}$. Therefore the equation has two roots

y	$=$	2	1	3	4
$\frac{36}{y}$	$=$	18	36	12	9
$25 - y^2$	$=$	21	24	16	9.

Hence, the leaſt root lies between one and two, and the greateſt root is 4. Therefore, ſince

$$a = c^m + d^m + \frac{c^m\,d^n - d^m\,c^n}{c^n - d^n}$$

$$25 = 4^2 + d^2 + \frac{4^2\,d - 4\,d^2}{4 - d} = 4^2 + d^2 + 4\,d$$

$$\therefore d^2 + 4\,d = 9$$

$$\therefore d = \sqrt{9+4} - 2 = \sqrt{13} - 2 = 1,60551\ldots$$

Let $9x^2 - x^3 = 100$ *.

$$M = \sqrt[m]{\frac{na}{m+n}} = \frac{2a}{3} = \frac{18}{3} = 6$$

$$M = \sqrt[m+n]{k \times \frac{n}{m}} = \sqrt[3]{100 \times \frac{2}{1}} = \sqrt[3]{200} = 5, \ldots$$

Hence, the equation has two roots

$$\text{Let } 9 - x = 4.$$

$$\frac{100}{9-x} = 25$$

$$x^2 = 25.$$

Hence, $x = 5$, and the greatest root is not a whole number.

$$a = c^m + d^m + \frac{c^m d^n - d^m c^n}{c^n - d^n} = c + d + \frac{cd^2 - dc^2}{c^2 - d^2}$$

$$= c + d - cd \times \frac{c-d}{c^2 - d^2} = c + d - \frac{cd}{c+d} = \frac{c^2 + cd + d^2}{c+d}$$

$$= \frac{c^2 + 5c + 25}{c+5} = 9$$

$$\therefore c^2 + 5c + 25 = 9c + 45$$

$$\therefore c^2 - 4c = 20$$

$$\therefore c = \sqrt{20+4} + 2 = 2 + \sqrt{24}.$$

Let $1000x - x^3 = 174$ †.

The least root is evidently less than one; therefore x^3 bears a very small proportion to $1000x$, and

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* Prob. XIII. XIV. Raphson. Analysis.

† Raphson. XXXI.

1000

$$1000\,d = 174 \text{ nearly}$$

$$\therefore d = {,}174;$$

$$\text{but } d = {,}174 + \frac{x^3}{1000}$$

$$\therefore d = {,}174 + \overline{\frac{174}{1000}}\Big|^3 \times \frac{1}{1000} \text{ nearly}$$

$$\text{Log. } \overline{174}|^3 = 3.\ \text{Log. } 174 = 3 \cdot 2{,}2405492 = 6{,}7216476$$

$$\therefore \overline{174}^3 = 5{,}268022 \times 10^6 \text{ nearly}$$

$$\therefore \frac{\overline{174}^3}{\overline{1000}|^3} \times \frac{1}{1000} = 5{,}268022 \times \frac{10^6}{10^9 \times 10^3} = \frac{5{,}268022}{10^6}$$

$$= {,}000005268022$$

$$\therefore d = {,}174005268022 + z.$$

For the greatest root a very near approach may be made in a similar manner.

$$1000\,c - c^3 = 174$$

$$\therefore 1000\,c - 174 = c^3$$

$$\therefore 1000 - \frac{174}{c} = c^2$$

$$\therefore c = \sqrt{1000} \text{ nearly}$$

$$\therefore c^2 = 1000 - \frac{174}{\sqrt{1000}} \text{ nearly}$$

$$\text{Log. } \frac{174}{\sqrt{1000}} = \left.\begin{matrix}\text{Log. } 174 - \tfrac{1}{2} \cdot 3 = 2{,}2405492 \\ -\ 1{,}5\end{matrix}\right\} =$$

$$= {,}7405492$$

$$\therefore \frac{174}{\sqrt{1000}} = 5{,}502362$$

∴

$$\therefore c^2 = 1000 - 5{,}502362 = 994{,}497637 \ldots$$
$$\therefore c = 31{,}5363 \text{ nearly.}$$

Let $77284\,x - x^3 = 8013128$. (RAPHSON, XI.)

Make $x = 100\,y$.

$$\therefore 7{,}7284\,y - y^3 = 8{,}013128$$

$$M = \sqrt{\frac{7{,}7284}{3}} = \sqrt{2{,}57613} = 1{,}605$$

$$y = \quad 1 \quad 1{,}2 \quad 1{,}4 \quad 18{,}3$$

$$\frac{8{,}013128}{y} = 8{,}013128 \quad 6{,}677 \quad 5{,}7236 \quad 4{,}378.$$

$$7{,}7284 - y^2 = 6{,}7284 \quad 6{,}2884 \quad 5{,}7684 \quad 4{,}3795.$$

The leaſt root appears to be between 1,2 and 1,4, but neareſt to 1,4 and nearly equal to 1,37; for 6,677 — 6,2884 = ,389 and 5,7236 — 5,7684 = ,044. $\therefore \frac{,044 + ,389}{20} = \frac{,433}{20} = ,0216$ and $\frac{,044}{,0216} = 2, \ldots$ Therefore ſomewhat more than two hundred parts are to be taken from 1,4, to give the true value of d. Since d = 1,37 nearly, $M - d = ,23 \ldots \therefore M + ,23 = 1,835 =$ c nearly. Hence, 1,83 was made a divider, and it appears that c is ſomewhat greater than 1,83. Therefore for the firſt approach make $c = 1{,}83 + z$. Raphſon after a very long calculation makes x to be 179,79652, an error which aroſe from inattention to the marks + and —, an inattention, which is very frequently fatal in approaches to the roots of an equation in this form. This equa-

equation alſo Baron Maſeres ſolved by tabular ſines, which give for the root c . 179,7368 *. But on the error having been pointed out to him, he ſolved it immediately by the method of approach, and confirmed the truth of this ſtatement.

Let $300\,x - x^3 = 1000$. (Raphson, XII.)

$$300\,d = 1000 + d^3 = 1000 \text{ nearly}$$

$$\therefore d = \frac{1000}{300} = \frac{10}{3} \text{ nearly}$$

$$\therefore d = \frac{10}{3} + \frac{10^3}{3^3 \cdot 300}$$

$$\therefore d = \frac{10}{3} + \frac{1000}{27 \cdot 300} = \frac{10}{3} + \frac{10}{3} \times \frac{1}{27} = \frac{10}{3} \times \overline{1 + \frac{1}{27}} = \frac{10}{3} \times \frac{28}{27} = \frac{280}{9 \cdot 9} = \frac{31,111}{9} = 3,45679012 \ldots$$

$$\therefore d = \frac{10}{3} + \frac{\overline{3,45679.}|^3}{300} = \frac{10}{3} + \frac{41,30665}{300} = 3,333 + ,13768883 = 3,47102216$$

$$\therefore d = \frac{10}{3} + \frac{\overline{3,471022}|^3}{300} = \frac{10}{3} + \frac{41,81884}{300} = 3,333 + ,3939613 = 3,47272946.$$

In the ſame manner approaches might be continually made, but they are very ſlow, ſince x^3 does not bear a

* Maſere's Diſſertation on the Uſe of the Negative Sign, page 255.

ſmall

ſmall proportion to 1000. Let this equation be tried by dividers.

$$M = \sqrt{\frac{300}{3}} = \sqrt{100} = 10$$

$$\begin{array}{llllllll}
x & = 5 & 3 & 4 & 3{,}5 & 15 & 16 \\
\frac{1000}{x} & = 200 & 333, & 250 & 285{,}5 & 66{,}\dot{6} & 62{,}5 \\
300 - x^2 & = 275 & 291 & 284 & 287{,}75 & 75 & 44.
\end{array}$$

Hence, d is between 3,5 and 4, and is equal to $3{,}47 + z$ and c is between 15 and 16, and is equal to $15{,}3 + z$. In the former caſe, $d = 3{,}472963553338$, in the latter, $c = 15{,}3208887 \ldots$ nearly

Let $5x - x^3 = 4$.

Here d is evidently equal to one. Therefore a which is equal to $c^2 + cd + d^2$ is equal to $c^2 + c + 1$.

$$\therefore 5 = c^2 + c + 1$$

$$\therefore c^2 + c = 4$$

$$\therefore c = \sqrt{4 + \tfrac{1}{4}} - \tfrac{1}{2} = \frac{\sqrt{17} - 1}{2}\text{*}$$

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* App. page 327.

CHAP.

CHAP. IV.

EQUATIONS OF THE THIRD CLASS.

EQUATIONS of the third claſs have on the unknown ſide three terms; and in one of their forms may have three roots, in others two roots, and in others can have only one root.

Forms of this Claſs.

$$\text{I.}\begin{cases} 1 \,.\, x^m + a\,x^n + b\,x^o = c \\ 2 \,.\, x^m + a\,x^n - b\,x^o = c \\ 3 \,.\, x^m - a\,x^n - b\,x^o = c \end{cases}$$

$$\text{II.}\begin{cases} 1 \,.\, a\,x^n + b\,x^o - x^m = c \\ 2 \,.\, a\,x^n - b\,x^o - x^m = c \\ 3 \,.\, b\,x^o - a\,x^n - x^m = c \end{cases}$$

$$\text{III.} \qquad x^m + b\,x^o - a\,x^n = c$$

The equations in the firſt order of forms of this claſs correſpond with thoſe of either the firſt or ſecond forms of equations of the ſecond claſs, and conſequently can have but one root. The ſecond order of equations correſponds with the third form of equations of the ſecond claſs, and conſequently equations of either of the forms in this order may have two roots. The third order has only one form, and this form correſponds, in one caſe, with the ſecond form of the ſecond claſs, and conſequently

quently can have but one root; in the other caſe it correſponds with the third form of equations of the ſecond claſs, in which conſequently it may have two roots. And, as both caſes take place ſometimes in the ſame equation, this form admits of three roots.

To make theſe propoſitions eaſier to the learner, they ſhall be firſt proved in equations of the ſimpleſt kinds in this claſs; namely, when m is equal to 3, n equal to 2, and o equal to one.

The equations of the firſt order become in this caſe,

$$1 .\ x^3 + a\,x^2 + b\,x = k$$
$$2 .\ x^3 + a\,x^2 - b\,x = k$$
$$3 .\ x^3 - a\,x^2 - b\,x = k.$$

1. $x^3 + a\,x^2 + b\,x = c$. This equation can evidently have only one root: for, if any number is added to or taken away from x, the ſum on the unknown ſide is made greater or leſs than it was before, and conſequently muſt be greater or leſs than k.

2. $x^3 + a\,x^2 - b\,x = k$. Since $x^3 + a\,x^2$ is greater than $b\,x$, and by adding to x, the increaſe of $x^3 + a\,x^2$ is greater than that of $b\,x$, the difference of the terms on the unknown ſide is increaſed by adding to x, and for the ſame reaſon is decreaſed by taking away from x. Conſequently by adding to x, the difference is made greater than k, by taking away from x the difference is made leſs than k.

3.

3 . $x^3 - ax^2 - bx = k$. In this caſe the increaſe of x^3 is greater than that of $ax^2 + bx$; therefore by adding to x the difference of the terms is made greater than k, and by taking from x on the ſame principle the difference is made leſs than k.

The ſecond order of equations in this claſs becomes, when m, n and o are reſpectively equal to 3, 2 and 1,

$$1.\ ax^2 + bx - x^3 = k$$
$$2.\ ax^2 - bx - x^3 = k$$
$$3.\ bx - ax^2 - x^3 = k.$$

1. $ax^2 + bx - x^3 = k$. In an equation of this form the unknown ſide does not always increaſe with the increaſe of x, nor decreaſe with its decreaſe.

Let x become $x + \dot{x}$, $\dot{x}$ being a very ſmall number, then

$$\left.\begin{array}{r} ax^2 \\ +bx \\ -x^3 \end{array}\right\} = \begin{array}{l} ax^2 + 2ax\dot{x} + \dot{x}^2 \\ +bx + b\dot{x} \\ -x^3 - 3x^2\dot{x} - 3x\dot{x}^2 - \dot{x}^3 \end{array} = \left\{\begin{array}{l} ax^2 + 2ax\dot{x} \\ +bx + b\dot{x} \text{ nearly} \\ -x^3 - 3x^2\dot{x}. \end{array}\right.$$

$\therefore\ 2ax\dot{x} + b\dot{x} \backsim 3x^2\dot{x}$ is the increaſe or decreaſe nearly of the unknown ſide of the equation. If $2ax + b$ is greater than $3x^2$, the equation is increaſed; if $2ax + b$ is leſs than $3x^2$, the equation is decreaſed.

Now let $\dot{x}$ be taken from x, or x be made $x - \dot{x}$, then

$$\left.\begin{array}{r} ax^2 \\ +bx \\ -x^3 \end{array}\right\} = \begin{array}{l} ax^2 - 2ax\dot{x} + \dot{x}^2 \\ +bx - b\dot{x} \\ -x^3 + 3x^2\dot{x} - 3x\dot{x}^2 + \dot{x}^3 \end{array} = \left\{\begin{array}{l} ax^2 - 2ax\dot{x} \\ +bx - b\dot{x} \text{ nearly} \\ -x^3 + 3x^2\dot{x} \end{array}\right.$$

Hence, $3x^2\dot{x} \backsim \overline{2ax\dot{x}+b\dot{x}}$ is the increaſe or decreaſe nearly of the equation. If $3x^2$ is greater than $2ax+b$, the equation is increaſed; if $3x^2$ is leſs than $2ax+b$, the equation is decreaſed. As long therefore as $2ax+b$ is greater than $3x^2$, the unknown ſide of the equation is increaſed by adding to x and decreaſed by taking away from x; and if it is equal to k in this ſtate, it muſt be equal to k again; for as it increaſes till $2ax+b$ is equal to $3x^2$, from that time the unknown ſide decreaſes by adding to x. And it conſtantly decreaſes from a number greater than k to nothing.

2. $ax^2 - bx - x^3 = k$. In this the unknown ſide increaſes or decreaſes by the term $2ax\dot{x} \backsim \overline{b\dot{x}+3x^2\dot{x}}$ when $\dot{x}$ is added to x, or by the term $\overline{b\dot{x}+3x^2\dot{x}} \backsim 2ax\dot{x}$, if $\dot{x}$ is taken from x, and the reaſoning is as before.

3. $bx - ax^2 - x^3 = k$. In this form the unknown ſide increaſes or decreaſes by the term $\overline{b\dot{x}+2ax\dot{x}} \backsim 3x^2\dot{x}$ by adding to x or by the term $\overline{2ax\dot{x}+3x^2\dot{x}} \backsim b\dot{x}$ when $\dot{x}$ is taken from x, and the reaſoning is the ſame as above.

III. $x^3 + bx - ax^2 = k$.

In this form the unknown ſide may firſt increaſe by adding to x, then decreaſe, and laſtly be capable of increaſe without limit. As before, it increaſes if $3x^2\dot{x} + b\dot{x}$ is greater than $2ax\dot{x}$, it decreaſes when $3x^2\dot{x} + b\dot{x}$ is leſs than $2ax\dot{x}$, and as x may be increaſed without

without limit, $3x^2\dot{x} + b\dot{x}$ muſt, by continually adding to x, become again greater than $2ax\dot{x}$ or equal to any aſſignable number. Hence, if the unknown ſide is equal to k when $3x^2\dot{x} + bx\dot{x}$ is greater than $2ax\dot{x}$, it may again become equal to k when $2ax\dot{x}$ is greater than $3x^2\dot{x} + bx\dot{x}$, and again equal to k when $3x^2\dot{x} + bx\dot{x}$ becomes again greater than $2ax\dot{x}$.

But an equation in this form cannot have more than three roots, for in either ſtate whether $2ax\dot{x}$ is greater or leſs than $3x^2\dot{x} + bx\dot{x}$, the unknown ſide conſtantly increaſes or decreaſes between the limits $x = o$ or x being ſuch that $3x^2\dot{x} + bx\dot{x} = 2ax\dot{x}$.

Let d, e, f be the roots of an equation in this form, d being greater than e, and e greater than f.

$$\text{Since } x^3 + bx - ax^2 = k$$
$$d^3 + bd - ad^2 = k$$
$$\text{Alſo } e^3 + be - ae^2 = k.$$
$$\text{Therefore } d^3 - e^3 + b \times \overline{d - e} = a \times \overline{d^2 - e^2}$$
$$\therefore d^2 + ed + e^2 + b = a \times \overline{d + e}$$
$$\text{Alſo } d^2 + df + f^2 + b = a \times \overline{d + f}$$
$$\therefore d \times \overline{e - f} + e^2 - f^2 = a \times \overline{e - f}$$
$$\therefore d + e + f = a.$$

Hence, in equations of this form having three roots, a the copart of the ſecond power of x is equal to the ſum of the roots.

Since $d^2 + ed + e^2 + b = a \times \overline{d + e}$ by ſubſtituting for a its equal the ſum of the roots.

$$d^2 + ed + e^2 + b = \overline{d + e + f} \times \overline{d + e}$$
$$\therefore d^2 + ed + e^2 + b = d^2 + de + de + e^2 + df + ef$$
$$\therefore b = de + df + ef.$$

Hence b, the copart of the ſimple unknown number, is always equal to the ſums of the products of each pair of the roots.

Now $d^3 + bd - ad^2 = k$. Therefore by ſubſtituting for b and a their equals in terms of the roots

$$d^3 + \overline{de + df + ef} \times d - d^2 \times \overline{d + e + f} = k$$
$$\therefore d^3 + d^2 e + d^2 f + def - d^3 - d^2 e - d^2 f = k$$
$$\therefore def = k.$$

Hence the known term is always the product of the roots multiplied together.

If one of the roots of an equation in this form having three roots is known, the other two may be found by means of an equation of the ſecond order. For

$$d^2 + ed + e^2 + b = a \times \overline{d + e}$$
$$\text{Alſo } d^2 + fd + f^2 + b = a \times \overline{d + f}$$
$$\text{and } e^2 + ef + f^2 + b = a \times \overline{e + f}.$$

Hence, if the leaſt root f is known, to find the next root e, the equation

$$e^2 + ef + f^2 + b = a \times e + f$$

would be employed.

And to find the greateſt root d, the equation

$$a^2 + df + f^2 + b = a \times \overline{d + f}$$

would be employed.

But they are both of the ſame form with the ſame coparts to the unknown number, and may be written thus:

$$x^2 + fx + f^2 + b = a \times \overline{f + x}$$

$$\therefore b + f^2 - af = ax - fx - x^2$$

$$\text{or } b + f^2 - af = x \times \overline{a - f} - x^2$$

Now this equation has two roots, whoſe ſum is equal to $a - f$ that is $d + e$ and their product is equal to $b + f^2 - af$ or to $b - f \times \overline{a - f}$ that is $b - f \times \overline{d + e}$ or $de + df + ef - df - ef$ that is to de.

Therefore d and e are the roots of the equation $x^2 + fx + f^2 + b = a \times \overline{f + x}$.

In the ſame manner if e had been known, the two roots d and f might be found from the equation

$$x^2 + ex + e^2 + b = a \times \overline{e + x}.$$

Or if d had been known, the other two roots might be found from the equation

$$x^2 + dx + d^2 + b = a \times \overline{d + x}.$$

Inſtance.

Inſtance.

Let $x^3 - 6x^2 + 11x = 6$ whoſe roots are 1, 2, 3. Suppoſe the leaſt root to have been the only one known. Then

$$x^2 + fx + f^2 + b = a \times \overline{f + x} \text{ becomes}$$

$$x^2 + x + 1 + 11 = 6 \times \overline{1 + x}$$

$$\therefore 12 - 6 = 6x - x - x^2$$

$$\therefore 6 = 5x - x^2$$

$$\therefore \sqrt{\frac{25}{4} - 6} = \begin{cases} \frac{5}{2} - x \\ \text{or } x - \frac{5}{2} \end{cases}$$

$$\text{but } \sqrt{\frac{25}{4} - 6} = \sqrt{\frac{25 - 24}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\therefore \frac{1}{2} = \frac{5}{2} - x$$

$$\therefore x = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$$

$$\text{or } \frac{1}{2} = x - \frac{5}{2}$$

$$\therefore x = \frac{5}{2} + \frac{1}{2} = \frac{6}{2} = 3.$$

The reader may try the ſame proceſs from either of the other two roots being ſuppoſed the only one known.

If the equation has only two roots, then e and f or d and e are equal. In the firſt caſe if d is known, the other root is found by making $a - d = 2e \therefore e = \frac{a - d}{2}$.

Or

Or if e in the ſecond caſe is known, $a - 2e = d$, which gives eaſily the number d. Thus, let

$$x^3 + 72x - 15x^2 = 112;$$

the roots are 7 and 4.

If 7 only had been known, then the root $e = \frac{a-d}{2} = \frac{15-7}{2} = 4$.

If 4 had been the root known,

$$d = a - 2e = 15 - 8 = 7.$$

In the ſame manner one root may be found from the other by means of the expreſſions b and k. For $k = def$, that is when $e = f$, $k = de^2$

$$\therefore d = \frac{k}{e^2}$$

$$\therefore e = \sqrt{\frac{k}{d}}$$

$$d = \frac{112}{16} = 7$$

$$e = \sqrt{\frac{112}{7}} = 4$$

Alſo $b = 2de + e^2$

$$\therefore d = \frac{b - e^2}{2e}$$

and $e = \sqrt{b + d^2} - d$

If the equation $x^3 + bx - ax^2 = c$ has only two roots d e, the equation

$$d^2 + e d + e^2 + b = a \times \overline{d + e}$$

has only one root, namely $\frac{a - d}{2}$ equal to e. For

$$d^2 + b - a d = a e - e d - e^2 = e \times \overline{a - d} - e^2$$

and ſince the two roots of the above equation are equal, $e = \frac{a - d}{2}$.

Subſtitute $\frac{a - d}{2}$ for e in the equation $d^2 + b - a d = e \times \overline{a - d} - e^2$.

$$\therefore d^2 + b - a d = \frac{a - d}{2} \times \overline{a - d} - \overline{\frac{a - d}{2}}\Big|^2 =$$

$$\overline{a^2 - 2 a d + d^2} \times \tfrac{1}{4}$$

$$\therefore 4 d^2 + 4 b - 4 a d = a^2 - 2 a d + d^2$$

$$\therefore 3 d^2 - 2 a d = a^2 - 4 b$$

$$\therefore d^2 - \frac{2 a d}{3} = \frac{a^2 - 4 b}{3}$$

$$\therefore d - \frac{a}{3} = \sqrt{\frac{a^2}{9} + \frac{a^2 - 4 b}{3}} = \sqrt{\frac{4a^2 - 4 b . 3}{9}}$$

$$\therefore d = \frac{a + 2 \sqrt{a^2 - 3 b}}{3}.$$

Alſo $e = \frac{a - d}{2} . \therefore e = \frac{1}{2} . a - \frac{a + 2 \sqrt{a^2 - 3 b}}{3}$

$$\therefore e = \tfrac{1}{2} \times \frac{2 a - 2 \sqrt{a^2 - 3 b}}{3} = \frac{a - \sqrt{a^2 - 3 b}}{3}$$

Hence, if the equation $x^3 + bx - a x^2 = c$ has only two roots as above deſcribed, the ſecond power of a muſt be greater than three times b.

Inſtance.

Instance.

Let the equation be $x^3 - 5x^2 + 7x = 3$ having two roots.

$$\therefore e = \frac{a - \sqrt{a^2 - 3b}}{3} = \frac{5 - \sqrt{25 - 21}}{3} = \frac{5 - 2}{3}$$

$$= 1 \text{ and } d = \frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{5 + 4}{3} = \frac{9}{3}$$

$= 3.$

Since the equation $e^2 + b - ae = x \times \overline{a - e} - x^2$ gives two values of x, which are the roots d and f, by making either of these values equal to e, the roots of the equation $x^3 - ax^2 + bx = k$, when it has only two roots, will be found. Thus

$$x \times \overline{a - e} - x^2 = e^2 + b - ae$$

$$\therefore x \backsim \frac{a - e}{2} = \sqrt{\overline{\frac{a - e}{2}}\Big|^2 - e^2 - b + ae} =$$

$$\frac{\sqrt{a^2 - 2ae + e^2 - 4e^2 - 4b + 4ae}}{2} =$$

$$\frac{\sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$$

$$\therefore x = \frac{a - e \pm \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$$

That is, $d = \dfrac{a - e + \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$

and $f = \dfrac{a - e - \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$

Now let the equation $x^3 - a x^2 + b x = k$ have only two roots, then d or f may be made equal to e. First let $d = e$.

$$\text{Then } e = \frac{a - e + \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$$

$$\therefore 3e - a = \sqrt{a^2 + 2ae - 3e^2 - 4b}$$

$$\therefore 9e^2 - 6ae + a^2 = a^2 + 2ae - 3e^2 - 4b$$

$$\therefore 12e^2 = 8ae - 4b$$

$$\therefore 2ae - 3e^2 = b$$

$$\therefore \frac{2ae}{3} - e^2 = \frac{b}{3}$$

$$\therefore \frac{a}{3} \backsim e = \sqrt{\frac{a^2}{9} - \frac{b}{3}}$$

$$\therefore e = \frac{a + \sqrt{a^2 - 3b}}{3}.$$

The other expression $\frac{a - \sqrt{a^2 - 3b}}{3}$ cannot be applied in this instance, since e being equal to d must be greater than $\frac{a}{3}$.

$$f = a - 2e = a - \frac{2a + 2\sqrt{a^2 - 3b}}{3} = \frac{a - 2\sqrt{a^2 - 3b}.}{3}$$

Hence if the equation has two roots, and the greater is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$, the smaller will be equal to $\frac{a - 2\sqrt{a^2 - 3b}}{3}$.

As

As f alſo may be equal to e, and f is equal to $\frac{a - e - \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$ then

$$e = \frac{a - e - \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}.$$

$$\therefore a - 3e = \sqrt{a^2 + 2ae - 3e^2 - 4b}$$

$$\therefore a^2 - 6ae + 9e^2 = a^2 + 2ae - 3e^2 - 4b$$

$$\therefore 8ae - 12e^2 = 4b$$

$$\therefore \frac{2ae}{3} - e^2 = \frac{b}{3}$$

$$\therefore e = \frac{a - \sqrt{a^2 - 3b}}{3}.$$

The other expreſſion $\frac{a + \sqrt{a^2 - 3b}}{3}$ cannot in this caſe be applied ſince a is greater than $3e$.

$d = a - 2e = a - \frac{2a - 2\sqrt{a^2 - 3b}}{3} = \frac{a + 2\sqrt{a^2 - 3b}}{3}$. Hence, if the equation has two roots, and the ſmaller root is equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$ the greater root muſt be equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$.

Thus, if the equation $x^3 - ax^2 + bx = k$ has only two roots, the one muſt be equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$, and

the other to $\frac{a - 2\sqrt{a^2 - 3b}}{3}$, or the one must be equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$, and the other $\frac{a + 2\sqrt{a^2 - 3b}}{3}$.

If the equation $x^3 + bx - ax^2 = c$ has three roots, e must be less than $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ but greater than $\frac{a - \sqrt{a^2 - 3b}}{3}$ and f less than $\frac{a - \sqrt{a^2 - 3b}}{3}$.

Instance.

$$x^3 - 12x^2 + 41x = 42$$

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{12 - \sqrt{144 - 123}}{3} = \frac{12 - \sqrt{21}}{3}$$

$$= \frac{12 - 4{,}5824\ldots}{3} = \frac{7{,}4175}{3} = 2{,}472.$$

The nearest whole number 2 being tried succeeds, and consequently there must be three roots.

$$\frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{12 + 2 \cdot 4{,}5824\ldots}{3} = \frac{12 + 9{,}1648}{3}$$

$$= \frac{12 + 9{,}1648}{3} = \frac{21{,}1648}{3} = 7{,}054\ldots$$

Hence e must lie between 2,472 and 7,054. Also it must be less than $\frac{a - f}{2}$ or $\frac{12 - 2}{2}$ or 5. Consequently 3 being the next whole number to 2,472 is tried and succeeds.

ceeds. The third root therefore being equal to $a - e - f$, or $12 - 3 - 2$ must be seven.

If the equations are of the forms admitting only of two roots, by applying the same mode of reasoning one root may be discovered from the other. Thus let the equation having two roots be of this form

$$a x^2 + b x - x^3 = k$$

$$\text{then } a d^2 + b d - d^3 = k$$

$$\text{and } a e^2 + b e - e^3 = k$$

$$\therefore a \times \overline{d^2 - e^2} + b \times \overline{d - e} = d^3 - e^3$$

$$\therefore a \times \overline{d + e} + b = d^2 + e d + e^2$$

$$\therefore a d^2 + a d e + b d = d^3 + e d^2 + e^2 d$$

$$\therefore a d^2 + b d - d^3 = e d^2 + e^2 d - a d e = d e \times \overline{d + e - a}$$

$$\text{But } a d^2 + b d - d^3 = k$$

$$\therefore k = d e \times \overline{d + e - a}.$$

Hence the sum of the roots must be greater than a.

$$\text{Since } a d^2 + b d - d^3 = k$$

$$b = \frac{k - a d^2 + d^3}{d} = \frac{d e \times \overline{d + e - a} - a d^2 + d^3}{d}$$

$$= e \times \overline{d + e - a} - a d + d^2 = e d + e^2 - a e - a d + d^2$$

$$\text{Let } 7 x^2 + 14 x - x^3 = 120.$$

One of the roots is 5.

$$k = d e \times \overline{d + e - a} = 5 d \times \overline{5 + d - 7} = 120$$

$$\therefore 25 d + 5 d^2 - 35 d = 120$$

$$\therefore d^2 - 2 d = 24$$

$$\therefore d = \sqrt{24 + 1} + 1 = 5 + 1 = 6.$$

Also

$$\text{Alſo } b = d \times \overline{d + e - a} - e \times \overline{a - e}$$

$$\therefore 14 = 6 \times \overline{6 + 5 - 7} - 5 \times 2 = 6 \times 4 - 10.$$

In the ſame manner the expreſſion for b and k are diſcovered in the form

$$a x^2 - b x - x^3 = k$$

$$\text{Thus } a d^2 - b d - d^3 = k$$

$$a e^2 - b e - e^3 = k$$

$$\therefore a \times \overline{d^2 - e^2} - b \times \overline{d - e} = d^3 - e^3$$

$$\therefore a \times \overline{d + e} - b = d^2 + e d + e^2$$

$$\therefore a d^2 + a d e - b d = d^3 + e d^2 + e^2 d$$

$$\therefore a d^2 - b d - d^3 = e d \times \overline{d + e} - a d e$$

$$\text{But } a d^2 - b d - d^3 = k$$

$$\therefore k = e d \times \overline{d + e - a}.$$

$$\text{Alſo } b = \frac{a d^2 - d^3 - k}{d} = \frac{a d^2 - d^3 - e d \times \overline{d + e - a}}{d}$$

$$= a d - d^2 - e d - e^2 + a e = e \times \overline{a - e} - d \times \overline{d + e - a}.$$

Hence, in this form alſo the ſum of the roots is greater than a.

For the form $b x - a x^2 - x^3 = k$ the inveſtigation is of the ſame kind.

$$b d - a d^2 - d^3 = k$$

$$b e - a e^2 - e^3 = k$$

$$\therefore b \times \overline{d - e} - a \times \overline{d^2 - e^2} = d^3 - e^3$$

$$\therefore b - a \times \overline{d + e} = d^2 + e d + e^2$$

$$\therefore b d - a d^2 - a d e = d^3 + e d^2 + e^2 d$$

∴

$$\therefore b\,d - a\,d^2 - d^3 = e\,d \times \overline{e + d + a}$$

$$\text{But } b\,d - a\,d^2 - d^3 = k$$

$$\therefore k = e\,d \times \overline{e + d + a}$$

$$\text{Alſo } b = \frac{k + a\,d^2 + d^3}{d} = \frac{e\,d \times \overline{e + d + a} + a d^2 + d^3}{d}$$

$$= e^2 + e\,d + a\,e + a\,d + d^2 = d \times \overline{a + d + e} + e \times \overline{a + e}.$$

Hence in this form a does not depend at all on the ſum of the roots, and the equation may be made of any two numbers whatever for roots. In either of the forms alſo if one root is found the other may be found by means of the expreſſions for k and b in an equation of two terms.

If there is only one root to equations in theſe forms, it may be diſcovered by means of an equation of the ſecond claſs. For in this caſe d and e become equal, and for the equations $a\,x^2 + b\,x - x^3 = k$

$$b = d \times \overline{d + e - a} - e \times \overline{a - e}$$

$$\text{That is, if } d = e$$

$$b = d^2 + d^2 - a\,d - a\,d + d^2$$

$$\therefore b = 3\,d^2 - 2\,a\,d$$

$$\therefore \frac{b}{3} = d^2 - \frac{2\,a\,d}{3}$$

$$\therefore d = \sqrt{\frac{a^2}{9} + \frac{b}{3}} + \frac{a}{3} = \frac{a + \sqrt{a^2 + 3\,b}}{3}.$$

Hence, if an equation in this form has two roots, the greateſt muſt be greater than $\frac{a + \sqrt{a^2 + 3\,b}}{3}$ and the

leaſt

least must be less than $\frac{a + \sqrt{a^2 + 3b}}{3}$.

In the equation $a x^2 - b x - x^3 = k$

$$b = e \times \overline{a - e} - d \times \overline{d + e - a}$$

That is, if $d = e$

$$b = a d - d^2 - d^2 - d^2 + a d = 2 a d - 3 d^2$$

$$\therefore \frac{b}{3} = \frac{2 a d}{3} - d^2$$

$$\left.\begin{array}{r}\therefore \frac{a}{3} - d = \\ \text{or } d - \frac{a}{3} = \end{array}\right\} \sqrt{\frac{a^2}{9} - \frac{b}{3}} = \frac{\sqrt{a^2 - 3b}}{3}$$

$$\therefore d = \frac{a \pm \sqrt{a^2 - 3b}}{3}$$

Since $2d$ is greater than a, d is greater than $\frac{a}{2}$, much more therefore greater than $\frac{a}{3}$, or a number less than $\frac{a}{3}$. But $\frac{a - \sqrt{a^2 - 3b}}{3}$ is less than $\frac{a}{3}$; and consequently this expression cannot be used for d in the given equation: and if there is only one root, that root is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$.

Hence, if the equation has two roots, the greatest root is greater than $\frac{a + \sqrt{a^2 - 3b}}{3}$ and the least root is less than $\frac{a + \sqrt{a^2 - 3b}}{3}$.

In

In the equation $b\,x - x^2 - x^3 = k$

$$b = d \times \overline{a + d + e} + e \times \overline{a + e}$$

That is, if $d = e$

$$b = a\,d + d^2 + d^2 + a\,d + d^2 = 3\,d^2 + 2\,a\,d$$

$$\therefore \frac{b}{3} = d^2 + \frac{2\,a\,d}{3}$$

$$\therefore d = \sqrt{\frac{a^2}{9} + \frac{b}{3}} - \frac{a}{3} = \frac{\sqrt{a^2 + 3\,b} - a}{3}$$

Hence, if the equation has two roots the one is greater and the other less than $\dfrac{\sqrt{a^2 + 3\,b} - a}{3}$.

The enquiry into the roots of any equations in these forms is thus confined within certain limits, which, with the limits prescribed by the equation itself, render the discovery of the roots oftentimes very easy.

Let $7\,x^2 + 14\,x - x^3 = 120$.

$$\therefore x \times \overline{7\,x + 14 - x^2} = 120$$

$\therefore 7\,x + 14$ is greater than x^2

Also $d + e$ is greater than 7

And d is greater than $\dfrac{a + \sqrt{a^2 + 3\,b}}{3}$ or $\dfrac{7 + \sqrt{49 + 42}}{3}$ or $\dfrac{7 + \sqrt{91}}{3}$ or $\dfrac{7 + 9, \ldots}{3}$ or $5, \ldots.$ and e is less than $5, \ldots.$

But d must be less than 7, for if $x = 7$ then $7\,x - x^2 + 14 = 14$ and 7×14 is only 98. Consequently

 the

the greater root muſt be between 5, ... and 7, and the leaſt root conſequently between 4 and 5, ... Hence, 6 and 5 are naturally the two firſt numbers to be tried, and they both ſucceed.

$$x = 6 \quad 5$$
$$\frac{120}{x} = 20 \quad 24$$
$$7x + 14 - x^2 = 20 \quad 24$$

Let $16x^2 - 31x - x^3 = 264$.

$$\therefore x \times \overline{16x - 31 - x^2} = 264$$

$d + e$ is greater than 16

$16x$ is greater than $31 + x^2$

$$d < \frac{a + \sqrt{a^2 - 3b}}{3} \text{ or } \frac{16 + \sqrt{256 - 93}}{3} \text{ or } \frac{28}{3} \text{ or}$$

9, ... and e is leſs than 9, ...

Hence, d is between 15, ... and 9, ... and e between 5, ... and 9, ...

$$x = 11 \quad 8$$
$$\frac{264}{x} = 24 \quad 33$$
$$16x - 31 - x^2 = 24 \quad 33.$$

Let $513x - 50x^2 - x^3 = 1062$.

$$\therefore x \times \overline{513 - 50x - x^2} = 1062$$

$\therefore$ 513 is greater than $50x + x^2$

$\therefore$ x is leſs than 8, ...

$$d < \frac{\sqrt{a^2 + 3b} - 50}{3} \text{ or } \frac{\sqrt{2500 + 1539} - 50}{3} \text{ or}$$

63,

$\frac{63,\ldots - 50}{3}$ or 4,3 and e is leſs than 4,3

$$x = 6 \quad 3$$

$$\frac{1062}{6} = 177 \quad 354$$

$$513 - 50x - x^2 = 177 \quad 354.$$

5 was not tried as the firſt divider becauſe $\frac{1062}{5}$ would have given a remainder, and $513 - 50x - x^2$ muſt, if x is a whole number, be a whole number.

The expreſſion for the root in equations of the forms admitting two roots when they have only one root may be diſcovered by another method. Since they are ſuppoſed to have only one root, that root muſt be ſuch as to make the unknown ſide the greateſt poſſible; and by adding to, or taking from the root, the unknown ſide is decreaſed. Conſequently, if M is that number which makes the unknown ſide equal to G or the greateſt poſſible, then M $+ y$ or M $- z$ being ſubſtituted for x will make the unknown ſide equal to the ſame number. Let

$$ax^2 + bx - x^3 = k.$$

$$\text{Then } aM^2 + bM - M^3 = G$$

$$\left.\begin{array}{r} ax^2 \\ +bx \\ -x^3 \end{array}\right\} \begin{array}{l} aM^2 + 2aMy + ay^2 \\ +bM + by \\ -M^3 - 3M^2y - 3My^2 - y^3 \end{array} \Bigg\} =$$

$$= \Bigg\{ \begin{array}{l} aM^2 - 2aMz + az^2 \\ +bM - bz \\ -M^3 + 3M^2z - 3Mz^2 + z^3 \end{array}$$

$$\therefore 2aM \times \overline{y+z} + b \times \overline{y+z} = 3M^2 \times \overline{y+z} + a \times \overline{z^2 - y^2} - 3M \times \overline{z^2 - y^2} + z^3 - y^3$$

∴

$$\therefore 2\,a\,M + b = 3\,M^2 + \frac{z^2 - y^2}{z + y} \times a - 3\,M \times \frac{z^2 - y^2}{z + y}$$
$$+ \frac{z^3 - y^3}{z + y}.$$

Now $\frac{z^2 - y^2}{z + y} \times a - 3\,M \times \frac{z^2 - y^2}{z + y} + \frac{z^3 - y^3}{z + y}$ muſt be equal to nothing, for z and y may be leſs than any aſſignable number as well as different aſſignable numbers.

$$\therefore 2\,a\,M + b = 3\,M^2$$
$$\therefore M^2 - \frac{2\,a\,M}{3} = \frac{b}{3}$$
$$\therefore M = \frac{a + \sqrt{a^2 + 3\,b}}{3}.$$

In the ſame manner for the other two forms M will be found by making $x = M + y$ and $M - z$, and the reſults from the ſubſtituting of theſe terms for x equal.

Hence, for the equation $a\,x^2 - b\,x - x^3 = k$.

$$2\,a\,M - 3\,M^2 = b$$

and for the equation $b\,x - a\,x^2 - x^3 = k$

$$b = 2\,a\,M + 3\,M^2.$$

Alſo for the form admitting three roots there will be a number between e and f which makes the unknown ſide greater than any number between e and f can do. Therefore $M + y$, or $M - z$ ſubſtituted for x in the equation, may make the unknown ſide equal to the ſame number.

$$\left.\begin{matrix} x^3 \\ -a x^2 \\ +b x \end{matrix}\right\} \left\{\begin{matrix} M^3 + 3M^2 y + 3My^2 + y^3 \\ -a M^2 - 2aMy - ay^2 \\ +bM + by \end{matrix}\right\} =$$

$$= \left\{\begin{matrix} M^3 - 3M^2 z + 3Mz^2 - z^3 \\ -aM^2 + 2aMz - az^2 \\ +bM - bz. \end{matrix}\right.$$

$$\therefore 3M^2 \times \overline{z+y} = 2aM \times \overline{z+y} - b \,.\times \overline{z+y}$$
$$+ 3M \times \overline{z^2 - y^2} - a \times \overline{z^2 - y^2} - z^3 + y^3$$

$$\therefore 3M^2 = 2aM - b + \frac{A - B}{z+y}$$

$$\therefore \frac{b}{3} = \frac{2aM}{3} - M^2$$

$$\therefore M = \frac{a \pm \sqrt{a^2 - 3b}}{3}$$

In this case the expression $\frac{a + \sqrt{a^2 - 3b}}{3}$ cannot be applied. For $\frac{a + \sqrt{a^2 - 3b}}{3}$ is greater than $\frac{a}{3}$. Therefore 2 M would be greater than $\frac{2a}{3}$. Consequently $a - 2M$ which is equal to d would be less than $\frac{a}{3}$ or the greater root less than the smaller, which is absurd. Hence, if the equation has only two roots, the least is equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$.

Also since $a - 2M = d$ and $M = \frac{a - \sqrt{a^2 - 3b}}{3}$

$$a - \frac{2a - 2\sqrt{a^2 - 3b}}{3} = d = \frac{a + 2\sqrt{a^2 - 3b}}{3}.$$

Hence, if the equation has two roots the greateſt root muſt be equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ and the leaſt muſt be equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$.

Alſo in this form there may be a number between d and e, making the unknown ſide leſs than any other number between d and e can do; and conſequently, if this number is called M, then $M + z$ and $M - y$ ſubſtituted for x may give equal reſults. Hence as before

$$\therefore 3M^2 + b = 2aM$$

$$\therefore \frac{b}{3} = \frac{2aM - M^2}{3}$$

$$\therefore \frac{\sqrt{a^2 - 3b}}{3} = M - \frac{a}{3} \text{ or } \frac{a}{3} - M$$

$$\therefore M = \frac{a \pm \sqrt{a^2 - 3b}}{3}$$

In this caſe $\frac{a - \sqrt{a^2 - 3b}}{3}$ cannot be applied for $a - 2M = f = a - \frac{2a - 2\sqrt{a^2 - 3b}}{3} = \frac{a + 2\sqrt{a^2 - 3b}}{3}$ a number greater than $\frac{a}{3}$, and conſequently

ſequently f the leaſt root would be greater than M, which is abſurd. Hence $M = \frac{a + \sqrt{a^2 - 3b}}{3}$.

Hence if the equation has three roots f is leſs than $\frac{a - \sqrt{a^2 - 3b}}{3}$, e is greater than $\frac{a - \sqrt{a^2 - 3b}}{3}$ but leſs than $\frac{a + \sqrt{a^2 - 3b}}{3}$, and d is greater than $\frac{a + \sqrt{a^2 - 3b}}{3}$.

If the equation has two roots, then one is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$ and the other is equal to $\frac{a - 2\sqrt{a^2 - 3b}}{3}$, or one is equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$ and the other is equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$.

If the equation has only one root, then no numbers $M - z$, $M + y$, can make the unknown ſide give equal reſults, and the expreſſions $a \pm \sqrt{a^2 - 3b}$ become impoſſible, or a^2 is either equal to or leſs than $3b$, in either of which caſes $\sqrt{a^2 - 3b}$ cannot expreſs any number whatſoever. But ſince the two roots of the equation $\frac{a + \sqrt{a^2 - 3b}}{3}$, $\frac{a - 2\sqrt{a^2 - 3b}}{3}$ may approach near-

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er to each other than by any assignable difference, they will approach nearer to $\frac{a}{3}$ than by any assignable difference, and the same may be said of the two roots $\frac{a - \sqrt{a^2 - 3b}}{3}$, $\frac{a + 2\sqrt{a^2 - 3b}}{3}$, that is, the equation will have only one root when it is equal to $\frac{a}{3}$ and $a^2 = 3b$, or if a^2 is less than $3b$ there can be only one root. There may be also only one root when a^2 is greater than $3b$, in which case the root must be greater than $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ or less than $\frac{a - 2\sqrt{a^2 - 3b}}{3}$.

Instances.

Let $x^3 - 12x^2 + 47x = 60$.

Here $\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{12 - \sqrt{144 - 141}}{3} = 4 - \frac{\sqrt{3}}{3} = 4 - 0, \ldots = 3, \ldots$

and $\frac{a + \sqrt{a^2 - 3b}}{3} = 4 + \frac{\sqrt{3}}{3} = 4 + 0, \ldots = 4, \ldots$

Hence,

Hence, if the equation has three roots, one muſt be leſs than 3, ... the next greater than 3, ... but leſs than 4, ... and the third muſt be greater than 4, ...

$$x \qquad = 3 \quad 4$$

$$\frac{60}{x} \qquad = 20 \quad 15$$

$$x^2 - 12x + 47 = 20 \quad 15.$$

Since 3 and 4 are roots, the third root is found by taking away 3 + 4 from 12. Or the third root is 5.

$$\text{Let } x^3 - 18x^2 + 105x = 200.$$

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{18 - \sqrt{324 - 315}}{3} = 6 - 1 = 5$$

$$\frac{a + \sqrt{a^2 - 3b}}{3} = 6 + 1 = 7$$

$$x \qquad = 5$$

$$\frac{200}{x} \qquad = 40$$

$$x^2 - 18x + 105 = 40.$$

Since 5 is a root, the equation can have only two roots and the ſecond root is equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ or $\frac{18 + 6}{3} = 8.$

Let $x^3 - 16x^2 + 77x = 98$.

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{16 - \sqrt{256 - 231}}{3} = \frac{16 - 5}{3}$$

$$= \frac{11}{3} = 2, \ldots .$$

$$\frac{a + \sqrt{a^2 - 3b}}{3} = \frac{16 + 5}{3} = 7$$

$$x = 7$$

$$\frac{98}{x} = 14$$

$$x^2 - 16x + 77 = 14.$$

Since one root is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$, the other must be equal to $\frac{a - 2\sqrt{a^2 - 3b}}{3} = \frac{16 - 10}{3}$ or $\frac{6}{3}$ or 2.

Let $x^3 - 12x^2 + 48x = 64$.

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{12 - \sqrt{144 - 144}}{3} = 4.$$

Here $a^2 = 3b$.

and 4 is therefore the only root.

Let $x^3 - 6x^2 + 20x = 15$.

Here $3b$ is greater than a^2, therefore the equation has only one root, and this root must be less than 6; for if x

is

is made 6 the unknown ſide becomes 120. And if $20x = 6x^2$ or $x = \frac{20}{6} = 3,3 \ldots.$ then x^3 is much greater than 15. Therefore the root muſt be leſs than 3,3. The numbers 2 and 1 therefore naturally preſent themſelves, and 1 is the root.

$$\text{Let } x^3 - 8x^2 + 20x = 400.$$

Here a^2 is greater than $3b$, but the root cannot be leſs than $\frac{a - \sqrt{a^2 - 3b}}{3}$ or $\frac{8 - 2}{3} = 2$, as is evident from inſpection: therefore the equation cannot have three roots. Alſo $\frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{8 + 4}{3} =$ which is evidently leſs than the root. Hence, this equation can have only one root, and this root muſt be greater than 8; for if $x = 8$ or $x^3 = 8x^2$ then the unknown ſide $= 20x = 160$, a number leſs than 400.

Since 10 is a divider of 400, it naturally preſents itſelf for trial.

$$x \qquad = 10$$

$$\frac{400}{x} \qquad = 40$$

$$x^2 - 8x + 20 = 40.$$

Hence 10 is the only root of this equation.

Let

Let $x^3 - 17\,x^2 + 54\,x = 350$.

$$\frac{a - \sqrt{a^2 - 3\,b}}{3} = \frac{17 - \sqrt{289 - 162}}{3} = \frac{17 - \sqrt{127}}{3}$$

$$= \frac{17 - 11,\ldots.}{3} = \frac{5,\ldots.}{3} = 16 \text{ which fails}$$

$$\frac{a + 2\sqrt{a^2 - 3\,b}}{3} = \frac{17 + 11,\ldots.}{3} = \frac{28,\ldots.}{3} =$$

9, ... which fails; for if x is 9, the unknown side is impossible.

x	$= 14$	15
$\frac{350}{x}$	$= 25$	$23, \ldots.$
$x^2 - 17x + 54 =$	12	$24.$

Hence x is between 14 and 15.

Let $x^3 - 65\,x^2 + 914\,x = 98746$.

$$\therefore z^3 - 6{,}5\,z^2 + 9{,}14\,z = 98{,}746$$

And $z \times \overline{z^2 - 6{,}5\,z + 9{,}14} = 98{,}746$.

If $z^3 = 6{,}5\,z^2$ or $z = 6{,}5$ then $9{,}14\,z$ will be less than 98,746, consequently z must be greater than 6,5.

$$\frac{a + 2\sqrt{a^2 - 3\,b}}{3} = \frac{6{,}5 + 2\sqrt{42{,}25 - 27{,}42}}{3} =$$

$$\frac{6{,}5 + 2\sqrt{14{,}83}}{3} = \frac{6{,}5 + 6,\ldots.}{3} = \frac{10,\ldots.}{3} = 3,\ldots$$

Consequently there can be only one root, and that greater from a preceding consideration than 6,5

$$z \qquad = 7 \qquad 7,2 \qquad 7,1$$

$$\frac{98,746}{7} \qquad = 14,\ldots 13,71 \quad 13,9$$

$$z^2 - 6,5\,z + 9,14 = 12,64 \quad 14,18 \quad 13,40.$$

Consequently $z = 7,15 + y$.

Here it may be obſerved that there is not much diffi-culty in finding the lower numbers; for $z^2 - 6,5\,z + 9,14$ is reſolvable into $\overline{z \times z - 6,5} + 9,14$. Conſe-quently when $z = 7,1$, the expreſſion is $7,1 \times \overline{7,1 - 6,5} + 9,14$ or $7,1 \times ,6 + 9,14 = 4,26 + 9,14 = 13,40$. By means of a book of logarithms the middle row might be continued without difficulty, and thus five or ſix fi-gures might be eaſily found for the root.

In ſearching after any one of the roots in the forms admitting two or three roots, the ſame methods are to be applied as if the equation had only one root. For within the limits, within which the root under examination lies, any addition to x makes the unknown ſide increaſe; any diminution from x makes it decreaſe. Conſequently the reaſoning uſed in forms admitting only one root is appli-cable to the other forms. Theſe forms are, as was ſaid before

$$x^3 + a\,x^2 + b\,x = k$$
$$x^3 - a\,x^2 - b\,x = k$$
$$x^3 + a\,x^2 - b\,x = k.$$

For the equation $x^3 + a\,x^2 + b\,x = k$ it is evident that x is leſs than $\sqrt[3]{k}$ or than $\sqrt{\frac{k}{a}}$ or than $\frac{k}{b}$, conſe-quently the leaſt of theſe numbers will be the neareſt li-mit,

mit, and by subſtituting this limit for x another leſs than x may be found.

Let $x^3 + 12x^2 + 42x = 261$.

$\therefore x \times \overline{x^2 + 12x + 42} = 261$

x is leſs than $\sqrt[3]{261}$ or 6, ...

x is leſs than $\sqrt{\frac{261}{12}}$ or $\sqrt{21}$, ... or 4, . . . ,

x is leſs than $\frac{261}{42}$ or $\frac{43,3}{7}$ or 6,19

x	$= 4$	3
$\frac{261}{4}$	$= 6,5,..87$. Therefore $x = 3$	
$x^2 + 12x + 42$	$= 106$	87.

Let $x^3 + 74x^2 + 8729x = 560783$.

$x = 10z$

$\therefore z^3 + 7,4z^2 + 87,29z = 560,783$

z is leſs than $\sqrt[3]{560.783}$ or 8, ...

z is leſs than $\sqrt{\frac{560,783}{7,4}}$ or 8, ...

z is leſs than $\frac{560,783}{87,29}$ or 6, ...

z	$= 5$	4	4,2	4,148
$\frac{560,783}{z}$	$= 112$	140,195	133,519	135,193
$z \times \overline{z + 7,4} + 87,29$	$= 149,29$	132,89	136,01	135,191104.

Hence

Hence z is very nearly equal to 4,148 or is equal to 4,148 + v.

Thus 4,148 + v might be used for the first approach to z, but the method of dividers points out a nearer number.

$$136{,}01 - 133{,}519 = 2{,}491$$
$$135{,}193 - 135{,}191104 = {,}002\ldots$$
$$\text{Sum} = 2{,}493$$

Difference between 4,148 and 4,2 = ,052 and $\frac{2{,}493}{{,}052}$ = 047 and ,047 × ,05 = ,00234 = ,002 .... nearly

$$\therefore z = 4{,}14805 \text{ nearly.}$$

Consequently for the first approach 4,14805 ± v might be used, or the process might be continued farther by the mode of dividers, thus:

Increment of $z^2 + 7{,}4z + 87{,}29 = 2z\dot{z} + 7{,}4\dot{z} + \dot{z}^2 =$

$$\dot{z} \times \overline{2z + 7{,}4 + \dot{z}}$$
$$z = 4{,}148 \text{ and } \dot{z} = {,}00005.$$

Increment of $z^2 + 7{,}4z + 8{,}729 =$

$\overline{8{,}296 + 7{,}4 + {,}00005} \times {,}00005 = {,}0007848025.$ To this add 135,191104 and $z^2 + 7{,}4z + 87{,}29$ by this increase from 4,141 to 4,1105 becomes 135,1918888025

$z = 4{,}14805$ 4,148051

$$\frac{560{,}783}{4{,}14805} = 135{,}19196$$

$z^2 + 7{,}4z + 8{,}729 = 135{,}1918888025$ 135,191900498601.

Hence z is evidently greater than 4,148051, and for the first approach 4,148051 + v might be used.

This

This number agrees with Raphson's solution, who after three approaches makes the root 4,1480514.

The advantage in using dividers is this. The lower series is easily found by deriving each term from the preceding by its increment, and when the root does not consist of a great number of figures, the division requisite for the middle series is not very troublesome. Also the number last found in the lower series serves in the first approach and saves a tedious multiplication. Thus if $z = 4{,}148051 + v$ or $a + v$

$$\therefore \overline{4{,}148051 + v} \times$$
$$\overline{a^2 + 7{,}4\,a + 8{,}729 + 2\,a\,v + v^2 + 7{,}4\,v} = 560783.$$

But $a^2 + 7{,}4\,a + 8{,}729$ has already been found to be equal to 135,191900498601

$$\therefore \overline{4{,}148051 + v} \times$$
$$\overline{135{,}191900498601 + 2\,a\,v + v^2 + 7{,}4\,v} = 560{,}783.$$

But by the method of dividers only two figures are gained by each trial, whereas by substitution when v is very small, the number of figures is frequently more than doubled.

In the first instance $x^3 + 12\,x^2 + 42\,x = 261$,* x was found to be less than 4, ... and as $x = \dfrac{261}{x^2 + 12\,x + 42}$ if x is less than 4, x must be greater than $\dfrac{261}{4^2 + 12 . 4 + 42}$ or $\dfrac{261}{16 + 48 + 42}$ or $\dfrac{261}{106}$ or 2, ...

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* Page 78.

Hence,

Hence x muſt lie between 2, ... and 4, ...

In general let $x^3 + a x^2 + b x = k$ let d be the leaſt number found which is greater than x. Then $\frac{k}{d^2 + a d + b}$ is leſs than x. Thus the firſt number to be uſed by the mode of dividers is confined within ſufficiently narrow limits.

$$\text{Let } x^3 - a x^2 - b x = k.$$

$$x^3 \text{ is greater than } a x^2 + b x$$
$$\therefore x^2 \text{ is greater than } a x + b.$$

$$\text{Let } a x + b = c^2.$$

$$\therefore x^2 \text{ is greater than } c^2$$
$$\therefore x \text{ is greater than } c.$$
$$\text{But } x^3 = k + a x^2 + b x$$
$$\therefore x \text{ is greater than } \sqrt{k + a c^2 + b c}$$
$$\text{Alſo } x^3 - k = a x^2 + b x$$
$$\therefore x \text{ is greater than } \sqrt[3]{k}$$
$$\text{Alſo } x \text{ is greater than } a \text{ and } x \text{ than } \sqrt{b}.$$

$$\text{Let } x^3 - 15 x^2 - 229 x = 525.$$

$$x \text{ is greater than } \sqrt[3]{525} \text{ or } 8, \ldots.$$
$$x \text{ is greater than } 15$$
$$x \text{ is greater than } \sqrt{229} \text{ or } 15, \ldots.$$
$$x^2 - 15 x \text{ is greater than } 229$$
$$\therefore x \text{ is greater than } 24, \ldots.$$

$$x \qquad = 25$$

$$\frac{525}{x} \qquad = 21$$

$$x^2 - 15x - 229 = 21.$$

$$\text{Let } x^3 - 10x^2 - 91x = 16.$$

x is greater than $\sqrt[3]{16}$ or 2, ...

x is greater than 10

x is greater than $\sqrt{91}$ or 9, ...

$x^2 - 10x$ is greater than 91 or 15, ...

$$x \qquad = 16 \quad 15{,}8 \qquad 15{,}817$$

$$\frac{16}{x} \qquad = 1 \quad 1{,}0126 \qquad 1{,}0115$$

$$x \times \overline{x - 10} - 91 = 5 \quad ,64 \qquad 1{,}007489.$$

$$\text{Hence } x = 15{,}8172 + z.$$

$$\text{Let } x^3 + ax^2 - bx = k.$$

Then $x^2 + ax$ is greater than b

$\therefore$ x is greater than $\dfrac{\sqrt{a^2 + 4b} - a}{2}$

$$x^3 = k + bx - ax^2$$

If bx is greater than ax^2 or b greater than a then x is greater than $\sqrt[3]{k}$.

But if b is lefs than a then x is lefs than $\sqrt[3]{k}$.

$$\text{Alfo } ax^2 = k + bx - x^3.$$

If b is greater than x^2 then x is greater than $\sqrt{\dfrac{k}{a}}$.

But

But if b is less than x^2, then x is less than $\sqrt{\frac{k}{a}}$.

Let $x^3 + 4x^2 - 17x = 12$.

x is greater than $\frac{\sqrt{16 + 78} - 4}{2}$ or 2, ... also x is greater than $\sqrt[3]{k}$ or 2, ... and x is greater than $\sqrt{\frac{k}{a}}$ or $\sqrt{\frac{12}{4}}$ or 1, ...

$$x \quad = 3$$
$$\frac{12}{3} \quad = 4 \quad \therefore x = 4$$
$$x^2 + 4x - 17 = 4$$

Let $x^3 + 22x^2 - 103x = 4$.

x is greater than $\sqrt{\overline{\frac{a}{2}}\Big|^2 + b} - \frac{a}{2}$ or $\sqrt{121 + 103}$ $= 14$, ... $-$ 11 or 3, ...

$$x \quad = 4$$
$$\frac{4}{x} \quad = 1 \quad \therefore x = 4$$
$$x^2 + 22x - 103 = 1.$$

Let $x^3 + 6x^2 - 183x = 2704$.

x is greater than $\sqrt{\frac{a^2}{4} + 183} - 3$ or 10, ...

x is greater than $\sqrt[3]{k}$ or $\sqrt[3]{1704}$ or 14, ...

x is less than $\sqrt{\frac{k}{a}}$ or $\sqrt{\frac{2704}{6}}$ or 21, ...

x

$$x \qquad = 15 \quad 16$$

$$\frac{2704}{x} \qquad = 180 \quad 169$$

$$x^2 + 6x - 183 = 132 \quad 169.$$

Therefore $x = 16$.

ANOTHER mode of discovering the number of roots in the equation $x^3 - a x^2 + b x = k$

x^3 may be taken, firſt leſs, then greater than $a x^2$.

If x is taken leſs than a, then the unknown ſide is leſs than $b x$; and if x is made equal to a then the unknown ſide is equal to $b x$. Conſequently, if $b a$ is equal to or leſs than k no number leſs than a can make the unknown ſide equal to k; and ſince the unknown ſide always increaſes, from the time that x is equal to a, by adding to x, the equation can have only one root. Hence if b is equal to or leſs than $\frac{k}{a}$ the equation has only one root.

When x is taken leſs than a,
$$a x^2 - x^3 = b x - k.$$

When x is taken greater than a
$$x^3 - a x^2 = k - b x.$$

In the firſt caſe, namely, when $a x^2 - x^3$ is equal to $b x - k$, let $b x - k = m x$, m being a variable number and

$$a x^2 - x^3 = m x$$
$$\text{or } a x - x^2 = m.$$

Now the ſide $a x - x^2$ firſt increaſes and then decreaſes;

creases; but m being equal to $b - \frac{k}{x}$ must as x increases always increase. $a x - x^2$ is greatest when $x = \frac{a}{2}$. Consequently, if in this case $a x - x^2$ is less than m, no number greater than $\frac{a}{2}$ can be substituted for x in this equation, and this equation admits only of one root. Hence the given equation $x^3 - a x^2 + b x = k$ cannot have more than two roots if $\frac{a^2}{4}$ is less than m. If $\frac{a^2}{4}$ is greater than m some number less than $\frac{a}{2}$ will make $a x - x^2$ equal to m, and then if the increase of $a x - x^2$ in that case is greater than m, some other greater number, but less than a, will make $a x - x^2$ equal again to m, and thus the equation $a x - x^2 = m$ will have two roots, and consequently the given equation $x^3 - a x^2 + b x = k$ will have three roots. Now m begins to exist at a certain value of x, namely, when $x = \frac{k}{b}$ and its increase is greatest at first, but grows continually less and less. Also the increase of $a x - x^2$ is greatest at first and constantly diminishes. Hence, m may become equal to $a x - x^2$ before x is equal to $\frac{a}{2}$, and then its increase may be less than that of $a x - x^2$, consequently the equation will have two roots. But if the increase of m is greater than that of $a x - x^2$, when m first becomes equal to $a x - x^2$, and is also greater when $x = \frac{a}{2}$, then there cannot be another root to the equation $a x - x^2 = m$,

m, and consequently the given equation will have only two roots. A few instances will make this subject very easy to the learner.

$$\text{Let } x^3 - 18\,x^2 + 105\,x = 200.$$

When x is less than 18, 105 x may be greater than 200. Consequently $a\,x - x^2 = m$.

Let $x = 9$ then $a\,x - x^2 = 81$ and $m = 105 - 22{,}2 = 82{,}7$. Hence the equation $a\,x - x^2 = m$ has only one root, and therefore the given equation has only two roots. By trying the expression $\frac{a - \sqrt{a^2 - 3\,b}}{3}$ or $\frac{18 - \sqrt{324 - 315}}{3}$ or 5 it appears that 5 is a root, and consequently the other root is $\frac{a + 2\sqrt{a^2 - 3\,b}}{3}$ or 8.

$$\text{Let } x^3 - 16\,x^2 + 77\,x = 98.$$

If x is less than 16, 77 x may be greater than 98, therefore $16\,x - x^2 = m$.

If $x = 8$, then $16\,x - x^2 = 64$, and $m = 77 - \frac{98}{8} = 77 - 12{,}25 = 64{,}75$.

Hence, the given equation has only two roots; one of which consequently is $\frac{a \pm \sqrt{a^2 - 3\,b}}{3}$ or $\frac{16 \pm \sqrt{256 - 231}}{3}$

or

or $\frac{16 + 5}{3}$ or 7 or 3, ... If 7 is one root, then $\frac{16 - 2.5}{3}$ or 2 is the other root, and either 7 or 2 being tried succeeds, consequently the two roots are 7 and 2.

$$\text{Let } x^3 - 12x^2 + 48x = 64.$$

If x is less than 12 then 48 x may be greater than 64, and if $x = 6$ then $12x - x^2 = 36$. But $m = 48 - \frac{64}{6} = 48 - 10{,}66 = 37{,}33$.

Hence it might be presumed that the equation has two roots; but on applying the expression $\frac{a \pm \sqrt{a^2 - 3b}}{3}$ it appears that $a^2 = 36$, and consequently this is the limiting case in which there is only one root $\frac{a}{3}$ or 4.

$$\text{Let } x^3 - 6x^2 + 20x = 15.$$

If x is less than 6, 20 x may be greater than 15, and if x is made equal to 3, then $6x - x^2 = 9$ and $m = 20 - \frac{15}{3} = 15$.

Hence, the equation has either three roots or only one. If there are three roots one must be greater than $\frac{6}{3}$ or 2; but 2 being tried is found to be greater than the root, consequently the equation can have only one root unity.

Let

Let $x^3 - 8x^2 + 20x = 400$.

If x is lefs than 8, $20x$ cannot be greater than 400, therefore the equation can have only one root, which is ten.

Let $x^3 - 8x^2 + 20x = 13$.

If x is lefs than 8, $20x$ may be greater than 13, and if x is equal to 4, $8x - x^2 = 16$, and $m = 20 - \frac{13}{4} = 20 - 3{,}25 = 16{,}75$. Hence there is a prefumption that the equation has only two roots, which is increafed by a^2 being greater than $3b$; but 4 and 3 are greater than the root, confequently the equation has either three roots or one. But if it has three roots, one muft be greater than 2, ... but any number greater than 2 makes the unknown fide greater than 13, and confequently the equation can have only one root unity.

Let $x^3 - 12x^2 + 41x = 42$.

If x is lefs than 12, then $41x$ may be greater than 42, and if $x = 6$ then $12x - x^2 = 36$, and $m = 41 - \frac{42}{6} = 34$. Hence the equation has either three roots or one root; but if it has only one root that root muft be lefs than $\frac{a - 2\sqrt{a^2 - 3b}}{3}$ or greater than $\frac{a + 2\sqrt{a^2 - 3b}}{3}$, that is, it muft be lefs than $\frac{12 - 2\sqrt{144 - 123}}{3}$ or lefs than

than unity, which is evidently impoſſible, or it muſt be greater than $\frac{12+2\sqrt{144-123}}{3}$ or greater than 7. On trying the latter ſuppoſition, it appears that 7 is a root, and the equation therefore has two other roots which are eaſily found to be 2 and 3.

$$\text{Let } x^3 - 15x^2 + 63x = 50.$$

If x is leſs than 15 then $63x$ may be greater than 50, and if $x = \frac{15}{2}$, then $15x - x^2 = 56{,}25$ and $m = 63 - \frac{50}{\frac{15}{2}} = 63 - 6{,}66 = 56{,}33$. Hence the equation will have only two roots, or if it has three, two will be very near to each other. $\frac{a \pm \sqrt{a^2-3b}}{3} = \frac{15 \pm \sqrt{225-189}}{3} = \frac{15 \pm 6}{3} = 7$ or 3.

x	$=$	3	7	1
$\frac{50}{x}$	$=$	16,6	7,1	50
$x^2 - 15x + 63$	$=$	27	7	49.

On trying 3 it does not ſucceed; but 7 is very near to the root, conſequently two roots are nearly equal to 7, and the other root muſt be nearly equal to one. On trying unity it appears that it is ſomewhat leſs than the root.

This method of finding the number of roots is improved by comparing together the increaſe of $a x - x^2$ and m, when x is increaſed by a number $\dot{x}$.

By adding $\dot{x}$ to x, $a x - x^2$ becomes

$$a x + a \dot{x} - x^2 - 2 x \dot{x} - \dot{x}^2$$

take away $a x \qquad - x^2$

The increase is $a \dot{x} - 2 x \dot{x} - \dot{x}^2$ or $\dot{x} \times \overline{a - 2 x - \dot{x}}$

$$m = b - \frac{k}{x}$$

$$m \text{ increased} = b - \frac{k}{x + \dot{x}}$$

$\therefore$ increase of $m = \frac{k}{x} - \frac{k}{x + \dot{x}} = \frac{k}{x} - \frac{k}{x} + \frac{k \dot{x}}{x^2} - \frac{k \dot{x}^2}{x^3} + \frac{k \dot{x}^3}{x^4} \ldots \mp \frac{k \dot{x}^{n-2}}{x^{n-1}} \pm \frac{k \dot{x}^{n-1}}{x^n} \times \frac{1}{x + 1} =$ $\frac{k \dot{x}}{x^2} - \frac{k \dot{x}^2}{x^3} + \frac{k \dot{x}^3}{x^4} \ldots\ldots\ldots \mp \frac{k \dot{x}^{n-2}}{x^{n-1}} \pm \frac{k \dot{x}^{n-1}}{x^n} \times \frac{1}{x + 1}$.

And this increase, if $\dot{x}$ is very small, is very nearly equal to $\frac{k \dot{x}}{x^2}$.

Consequently the increase of $a x - x^2$ is to the increase of m as $a - 2 x$ to $\frac{k}{x^2}$ nearly.

Let $x^3 - 8 x^2 + 20 x = 13$.

Here $m = 20 - \frac{13}{x}$ and begins from $\frac{13}{20}$, that is, m cannot be less than or equal to $\frac{13}{20}$. Hence unity is taken as the first value of x in comparing $a x - x^2$ with m, or $8 x - x^2$ with $20 - \frac{13}{x}$, and it appears that if $x = 1$ then

8

$8x - x^2 = 20 - \frac{13}{x} = 7$, consequently unity is a root of the proposed equation; and here the increase of m is greater than that of $ax - x^2$. If $\dot{x} = 1$, the increase of $m = 6,5$ and that of $ax - x^2 = 5$. If $\dot{x}$ again is equal to unity, since x is now equal to two, the increase of $ax - x^2$ is greater than that of m, but m is greater than $ax - x^2$ when $x = 2$, and is also greater than $ax - x^2$ when $x = 3$, but they are so nearly equal to each other in this last case, and the increase of $ax - x^2$ is so much greater than m that the trouble of comparing the farther process of the increase of each is greater than that of trying the equation by means of the discovered root unity. From this root, by means of the equation

$$x^2 + ex + e^2 + b = a \times \overline{x + e}$$

$$\text{or } x^2 + x + 1 + 20 = 8 \times \overline{x + 1}$$

$$\text{or } 13 = 7x - x^2$$

it appears, that there cannot be another root, for $\overline{\frac{7}{2}}\Big|^2$ is less than 13. But as $\overline{\frac{7}{2}}\Big|^2$ is nearly equal to 13, the reason of the above process being troublesome is apparent.

Let $x^3 - 18x^2 + 105x = 200$.

Here the limit of m is $\frac{200}{105}$ or 1, ... and when $x = 2$, $m = 5$ and $18x - x^2 = 32$; but when $x = 2$, the increase of m is to that of $ax - x^2$ as 50 to 14, and when $x = 5$, the increase of m is equal to that of $ax - x^2$. Consequently $ax - x^2$ is equal to m when x is either

equal

equal to five or nearly equal to five. On trying five, it appears that five is a root; but as the increase of m is again equal to that of $a x - x^2$, when x is some number between 5 and 9, there will be another root to the equation, and consequently the given equation, since it can have no root when x is equal to or greater than 18, can have but two roots.

$$\text{Let } x^3 - 16 x^2 + 77 x = 98.$$

The limit of m is $\frac{98}{77}$ or 1, ... Consequently 2 is tried for x, and it appears that $16 x - x^2$ is then equal to m or $77 - \frac{98}{2}$. The increase of $16 x - x^2$ is first less than that of m, and then greater and then less again, and they are equal when $x = 7$. Consequently the other root will be either 7 or nearly equal to 7, and as 7 succeeds there can be no other root.

One advantage attending this mode is, that it gives a learner a practical knowledge of equations, which will be very useful as he advances in science.

Equations of this form may be changed into equations of the second order, and thus admit frequently an easy mode of solution *.

$$x^3 - a x^2 + b x = k.$$

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* See Part I. page 139.

Let

$$\text{Let } x = \frac{a}{3} \pm z.$$

$$\left.\begin{array}{rl} \therefore\ x^3 & \left\{ \overline{\frac{a}{3}}\Big|^3 \pm 3 \cdot \overline{\frac{a}{3}}\Big|^2 z + 3\ \frac{a}{3} z^2 \pm z^3 \right. \\ -\,a\,x^2 & \left\{ -a \cdot \overline{\frac{a}{3}}\Big|^2 \mp 2 \cdot \frac{a\,a}{3} \cdot z - a\,z^2 \right. \\ +\,b\,x & \left\{ b \cdot \frac{a}{3} + b \cdot z \right. \end{array}\right\} = k.$$

In the new equation, one term is $z^2 \times \overline{\frac{3\,a}{3} - a}$, that is $z^2 \times \overline{a - a}$ or o. Hence, in any propoſed equation of this form, by making x equal to $\frac{a}{3} \pm z$, a new equation of the ſecond order is the reſult, in which there is not an unknown number raiſed to the ſecond power.

Inſtance.

$$x^3 - 3x^2 - 36\,x = 342.$$

$$\text{Let } x = \frac{a}{3} + y = 1 + y$$

$$\left.\begin{array}{rcl} \therefore\ x^3 & = & 1 + 3\,y + 3\,y^2 + y^3 \\ -\,3\,x^2 & = & -\,3 - 6\,y - 3\,y^2 \\ -\,36\,x & = & -\,36 - 36\,y \end{array}\right\} = 342$$

$$\therefore\ -38 - 39\,y + y^3 = 342$$

$$\therefore\ y^3 - 39\,y = 380$$

If y	$=$	8	9
$\frac{380}{y}$	$=$	47,5	42,22
$y^2 - 39$	$=$	25	42

∴ y

$$\therefore y = 9 + z$$

$$\left.\begin{array}{r} y^3 \\ -39y \end{array}\right\{ \begin{array}{l} 729 + 243z + 27z^2 + z^3 \\ -351 - 39z \end{array} \left.\right\} = 380$$

$$\therefore 378 + 204z + 27z^2 + z^3 = 380$$

$$\therefore 204z = 2 - 27z^2 - z^3$$

$$\therefore 68z = ,66\dot{6} - 9z^2 - \frac{z^3}{3}$$

$\therefore z = ,0097912$ and $y = 9 + z = 9,0097912$

But $x = 1 + y$ $\therefore x = 10,0097912$

$$\therefore x^3 = 1002,9402369665820200 54528$$

$$\left.\begin{array}{l} -3x^2 = -300,58775960279232 \\ -36x = -360,3524832 \end{array}\right\} = -$$

$$\text{Sum} = 660,94024280279232$$

$$\therefore x^3 - 3x^2 - 36x = 341,99999416378970054528$$

which is leſs than 342 by ,00000 583, &c.

The learner will ſee the advantage of this change by ſolving the original equation $x^3 - 3x^2 - 36x = 342$ in its preſent ſtate, and having diſcovered that x is nearly equal to ten, he will make it equal to $10 + z$, and then proceed by the uſual mode of approach.

By changing the equation in this manner the root may be diſcovered by a compariſon of the dividers of the known terms in the original and the new equation *.

Let $x^3 + 3x^2 + 2x = 24$.

x is a divider of 24, and the dividers of 24 are 1 2 3 4 6 8 12 24.

* See Part I. page 140.

The

The new equation is $y^3 - y = 24$.

y is therefore a divider of 24, and $y = x + 1$. Therefore, by adding unity to one of the dividers of 24, a number will be found which is alſo a divider of 24, and conſequently may be a root of the propoſed equation. On examination there are three numbers anſwering to this condition 1 2 3. Conſequently one of them muſt be the root of the propoſed equation; but if $x = 3$ then x^3 is alone greater than 24, and 3 cannot be a root. Unity alſo cannot evidently be the root. Therefore 2 is the root of the given equation.

Let $x^3 - 18x^2 + 105x = 200$.

Dividers 1 2 4 5 8 10.

Let $x = 6 \pm y$.

The new known term is either $200 - 6^3 + 18.6^2 - 105.6$, which is equal to 2, whoſe dividers are 1 and 2.

Hence 4 5 8 anſwer to the conditions, and they cannot all three be roots, for $4 + 5 + 8$ is not equal to 18, and 8 and 5 being roots there cannot be another root, for 6, which muſt be the root, if the equation has three roots, is not a divider of 200.

If x had been made equal to $y - 6$ the new known term would be $200 + 6^3 + 18.6^2 + 105.6$ or 1694, whoſe dividers are 1 2 7 11 14, and conſequently x is to be looked for among the numbers $7 - 6$, $11 - 6$, $14 - 6$ or 1, 5 and 8, and, as before, the two roots are found to be 5 and 8.

Let

Let $x^3 - 12x^2 + 48x = 64$.

The dividers of 64 are 1 2 4 8, the new known term is $64 \mp 64 + 12 \times 16 \mp 48 \times 4$ by making $x = 4 \pm y$, that is 0 or 512. Hence it is evident that $x = 4$.

This mode of ſolution can ſeldom be applied with advantage unleſs the copart of the ſecond term $a\, x^2$ is diviſible by three.

The rule of double poſition for the ſolution of equations is recommended by ſuch high authority that it deſerves particular examination, and to do this with greater juſtice, the rule ſhall be given in the recommender's words, together with his own example, to be compared with the ſame equation ſolved by the method of dividers.

" RULE.

" 1. FIND, by trial, two numbers, as near the true root as poſſible, and ſubſtitute them in the given equation inſtead of the unknown quantity; marking the errours which ariſe from each of them.

" 2. Multiply the difference of the two numbers, found by trial, by the leaſt errour, and divide the product by the difference of the errours, when they are alike, but by their ſum when they are unlike. Or ſay, As the difference or ſum of the errours is to the difference of the two numbers, ſo is the leaſt errour to the correction of its ſuppoſed number.

" 3. Add the quotient, laſt found, to the number belonging

ing to the leaſt errour, when that number is too little, but ſubtract it when too great, and the reſult will give the true root *nearly*.

"4. Take this root and the neareſt of the two former, or any other that may be found nearer; and by proceeding in like manner, a root will be had ſtill nearer than before; and ſo on to any degree of exactneſs required.

"EXAMPLE.

"To find the root of the equation $x^3 + x^2 + x = 100$, or the value of x in it.

"Here it is ſoon found that x lies between 4 and 5. Aſſume therefore theſe two numbers, and the operation will be as follows:

1ſt Sup.		2d Sup.
4	- - x - -	5
16	- - x^2 - -	25
64	- - x^3 - -	125
84	- ſums -	155
−16	- errours -	+55

the ſum of which is 71.
Then as 71 : 1 :: 16 : ·225.
Hence x = 4·225 nearly.

"Again, ſuppoſe 4·2 and 4·3, and repeat the work as follows:

1ſt Sup.		2d Sup.
4·2	- x -	4·3
17·64	- x^2 -	18·49
74·088	- x^3 -	79·507
95·928	- ſums -	102·297
−4·072	- errours -	+2·297

the ſum of which is 6·369
As 6·369 : 1 :: 2·297 : 0·036
This taken from - 4·300
leaves x nearly = 4·264

"Again, suppose 4·264 and 4·265, and work as follows:

4·264	-	x	-	4·265
18·181696	-	x^2	-	18·190225
77·526752	-	x^3	-	77·581310
99·972448	-	sums	-	100·036535
−0·027552	-	errours	-	+0·036535

the sum of which is ·064087.

Then as ·064087 : ·001 : : 4·264 : 0·0004299

To this adding - 4·264

gives x very nearly = 4·2644299."

The same example worked out by the mode of dividers.

$$\text{Let } x^3 + x^2 + x = 100.$$

$$\text{Therefore } x \times \overline{x^2 + x + 1} = 100$$

If x	=	4	5	4,26	4,27
$\frac{100}{x}$	=	25	20	23,4741	23,4192
$x^2 + x + 1$	=	21	31	23,4076	23,5029
Diff.	=	4	11	,0665	,0836

$$4 + 11 = 15 \quad \frac{15}{10} = 1,5 \text{ and } \frac{4}{1,5} = \frac{,8}{,3} = \frac{,266}{,1}$$

Hence 4,26 must be a nearer value of x, and on trying 4,26 it appears to be so near to x that 4,27 must be greater than x.

,0665

$$\frac{,0665 + ,0836}{10} = \frac{,1501}{10} = ,01501 \text{ and } \frac{,06650}{,01501} =$$
4,430.

Hence, x is equal to 4,26443 nearly.

By the mode of double poſition, the firſt ſtep gives 4,225, and to do this the diviſion of 16 by 71 to four places is neceſſary.

By the mode of dividers the firſt ſtep gives a nearer number for the root, namely, 4,26, and a diviſion by one figure only is neceſſary.

By the mode of double poſition, the ſecond ſtep gives 4,264; but to do this the numbers 4,2 4,3 muſt be raiſed to the third powers, and 2,297 muſt be divided by 6,369 as far as four places.

By the mode of dividers, the ſecond ſtep gives 4,26443, and to do this 100 muſt be divided by each of the numbers 426 and 427 to ſix places, and the numbers 4,26 and 4,27 muſt be raiſed to the ſecond powers or derived by the mode of increments more eaſily from the value of $x^2 + x + 1$ when $x = 4$.

By the mode of double poſition 4,264 and 4,265 muſt be raiſed to the third powers, and 4,264 is to be divided by ,064087; and when this operation is performed the reſult is very nearly the ſame with that obtained by the mode of dividers by two operations, each eaſier than the correſponding operation by the rule of double poſition.

Thus,

$$\begin{array}{lrl} \text{Thus, by double position} & x &= 4{,}2644299 \\ \text{by dividers} \quad - & x &= 4{,}2644303 \\ \hline \text{difference} \quad - \; - & &= 0{,}0000004 \end{array}$$

Hence it appears, that the mode of dividers is far preferable to that by double position, and the number 4,26443 having been obtained x should be made equal to 4,26443 $\pm z$, and by the method of approach a greater number of figures will be obtained than by either the rule of double position or the mode of dividers.

Another mode stands recommended by the authority of Dr. Halley, but it is probable that very few persons since his time have adopted it, and a very accurate investigation of it by Baron Maseres in several instances has proved its inferiority to the simple method of approach. Instead of rejecting all the powers of z in the new equation, as in the simple mode of approach, Dr. Halley retains the terms, having in them z^2. Thus let $x^3 - a x^2 + b x = c$.

$$\text{Make } x = d \pm z$$

$$\left.\begin{array}{r} \therefore x^3 \\ -a x^2 \\ +b x \end{array}\right\} \left.\begin{array}{l} d^3 \pm 3 d^2 z + 3 d z^2 \pm z^3 \\ -a d^2 \mp 2 a d z - a z^2 \\ + b d \pm b z \end{array}\right\} = k$$

By neglecting only the term z^3 there remains an equation of the second class to be solved, the copart of whose simple term z being $\frac{\pm 3 d^2 \mp 2 a d \pm b}{3 d - a}$, will, after the first operation, be a fraction with a considerable number of figures in both the upper and the lower parts. Now

Now half of this fraction is to be raised to the second power, and then the second root of the known term added to the new fraction is to be extracted, so that though a greater number of figures will be attained by each operation, yet the trouble of obtaining them far outbalances this advantage, and in practice it will be found much easier to use more operations by the simple mode of approach to arrive at the same number of figures.

CHAP. V.

EQUATIONS OF THE FOURTH CLASS.

EQUATIONS of the fourth class have four terms on the unknown side, and are divided into four orders. Those in the first order can have only one root, in the second only two, in the third three roots, and in the fourth four roots.

$$\text{I.}\left\{\begin{array}{l} 1.\ x^m + a x^n + b x^o + c x^p = k \\ 2.\ x^m + a x^n + b x^o - c x^p = k \\ 3.\ x^m + a x^n - b x^o - c x^p = k \\ 4.\ x^m - a x^n - b x^o - c x^p = k. \end{array}\right.$$

$$\text{II.}\left\{\begin{array}{l} 1.\ a x^n + b x^o + c x^p - x^m = k \\ 2.\ a x^n + b x^o - c x^p - x^m = k \\ 3.\ a x^n - b x^o - c x^p - x^m = k \\ 4.\ b x^o + c x^p - a x^n - x^m = k \\ 5.\ b x^o - c x^p - a x^n - x^m = k \\ 6.\ c x^p - b x^o - a x^n - x^m = k. \end{array}\right.$$

III.

$$\text{III.}\begin{cases}1.\ x^m - a\,x^n + b\,x^o + c\,x^p = k\\ 2.\ x^m + a\,x^n - b\,x^o + c\,x^p = k\\ 3.\ x^m - a\,x^n - b\,x^o + c\,x^p = k\\ 4.\ x^m - a\,x^n + b\,x^o - c\,x^p = k.\end{cases}$$

$$\text{IV.}\quad c\,x^p - b\,x^o + a\,x^n - x^m = k.$$

In equations of the firſt and third orders the terms on the unknown ſide are written down according to the indexes of the unknown powers $m\ n\ o\ p$. In the ſecond and fourth orders the terms are not written down in the ſucceſſion of the indexes, but the higheſt index is in the laſt term. In the firſt order there is either no change or only one change of the marks of adding and ſubtracting. In the third order there are either two or three changes of the marks of adding and ſubtracting. Hence, if an equation in theſe orders is written down agreeably to the indexes, the number of roots in the form to which it belongs is determined by obſerving the change in the marks: if there is either no change or one change the equation can have only one root: if there are either two or three changes (that is, if there is more than one change) the equation belongs to a form capable of having three roots.

In the ſecond order there are either one or two changes of the marks; in the fourth order there are three changes of the marks. Theſe changes are obſerved by reading the equation according to the indexes $m\ n\ o\ p$. Thus, in the equation

$$b\,x^o - c\,x^p - a\,x^n - x^m = k$$

by reading it in the order preſcribed $- x^m$, $- a\,x_n$, $+ b\,x^o$, $- c\,x^p$ the two firſt terms x^m and $a\,x^n$ have the ſame

ſame mark, the third term has a different mark, conſequently there is one change in reading the three terms x^m, $a x^n$, $b x^o$, and the two terms $b x^o$, $c x^p$ have different marks, conſequently there are two changes of marks by reading according to the order of the indexes. Hence, if in an equation of this claſs, the unknown term, with the higheſt index, is not the firſt term, the equation belongs to the ſecond or the fourth order. If there are either one or two changes of the marks, the equation is of the form capable of having two roots. If there are three changes of the marks, the equation is of the fourth order, and of a form capable of having four roots.

Let it be required to determine the number of roots in the equation $54 x^3 - 32 x^2 + 20 x^4 - 50 x = 200$.

The equation written in order is $20 x^4 + 54 x^3 - 32 x^2 - 50 x = 200$, conſequently the equation having only one change, and the higheſt power of the unknown number being in the firſt term, the equation can have only one root.

$$\text{Let } 30 x^2 - 12 x - 50 x^4 + 172 x^3 = 340.$$

$$\therefore 172 x^3 + 30 x^2 - 12 x - 50 x^4 = 340.$$

The term x^4 being laſt, and there being two changes in the marks, the equation cannot have more than two roots.

$$\text{Let } 20 x - 50 x^2 - 30 x^3 + 40 x^4 = 2012.$$

$$\therefore 40 x^4 - 30 x^3 - 50 x^2 + 20 x = 2012.$$

The term $40 x^4$ being the firſt, and there being two changes

changes of the marks, the equation cannot have more than three roots.

$$\text{Let } 70x^3 - 12x^2 + 50x - 30x^4 = 59.$$

$$\therefore 50x - 12x^2 + 70x^3 - 30x^4 = 59.$$

The term $30x^4$ being the laſt, and there being three changes of the marks, the equation is of the fourth order, and conſequently is of a form capable of having four roots.

To find the relation of the coparts and roots in the equation

$$cx - bx^2 + ax^3 - x^4 = k.$$

Let e, f, g, h be the roots. Then

$$ce - be^2 + ae^3 - e^4 = k$$

$$\text{and } cf - bf^2 + af^3 - f^4 = k$$

$$\therefore c \times \overline{e-f} - b \times \overline{e^2 - f^2} + a \times \overline{e^3 - f^3} = e^4 - f^4$$

$$\therefore c - b \times \overline{e+f} + a \times \overline{e^2 + ef + f^2} = e^3 + e^2f + ef^2 + f^3$$

$$c - b \times \overline{e+g} + a \times e^2 + eg + g^2 = e^3 + e^2g + eg^2 + g^3$$

$$\therefore ae \times \overline{f-g} + a \times \overline{f^2 - g^2} - b \times \overline{f-g} = e^2 \times \overline{f-g} + e \times \overline{f^2 - g^2} + f^3 - g^3$$

$$\therefore ae + a \times \overline{f+g} - b = e^2 + e \times \overline{f+g} + f^2 + fg + g^2$$

$$ae + a \times \overline{f+h} - b = e^2 + e \times \overline{f+h} + f^2 + fh + h^2$$

$$\therefore a \times \overline{g-h} = e \times \overline{g-h} + f \times \overline{g-h} + g^2 - h^2$$

$$\therefore a = e + f + g + h.$$

By the ſame proceſs as in pages 25, 52, it is proved that b is equal to the ſum of the products of each pair of roots,

roots, c the sum of the products of the roots taken by three's, k the product of all the roots.

If one of the roots of an equation in this order is known, the other three may be found by means of an equation of the third class. Let e be the root known, then the equation to find the roots is

$$c - b \times \overline{e + x} + a \times \overline{e^2 + ex + x^2} = e^3 + e^2 x + e x^2 + x^3.$$

Instance.

$$11208\, x - 1708\, x^2 + 78\, x^3 - x^4 = 16192$$

$$x = 1 \quad 2$$

$$\frac{16192}{x} = 16192 \quad 8096$$

$$11208 - 1708 x + 78\, x^2 - x^3 = 9577 \quad 8096.$$

Hence 2 is a root of the proposed equation.

$$\therefore 11208 - 1708 \times \overline{2 + x} + 78 \times \overline{4 + 2x + x^2} = 8 + 4x + 2\, x^2 + x^3$$

$$\therefore 11208 - 3416 + 312 - 8 = 1708\, x - 156\, x + 4\, x - 78\, x^2 + 2\, x^2 + x^3$$

$$\therefore 8096 = 1556\, x - 76\, x^2 + x^3$$

$$x = 10 \quad 8$$

$$\frac{8096}{x} = 809{,}6 \quad 1012$$

$$1556 - 76\, x + x^2 = 896 \quad 1012.$$

Hence 8 is another root of the proposed equation, and *

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* See page 53.

$$x^2 + 8x + 64 + 1556 = 76 \times \overline{8 + x}$$
$$1620 - 608 = 76x - 8x - x^2$$
$$\therefore 1012 = 68x - x^2$$
$$\therefore \sqrt{34^2 - 1012} = \begin{cases} 34 - x \text{ or} \\ x - 34 \end{cases}$$
$$\therefore 12 = \begin{cases} 34 - x \text{ or} \\ x - 34 \end{cases}$$
$$\therefore x = 46 \text{ or } 22.$$

Hence the roots of the propoſed equation are 2, 8, 22, 46 and

$$2 + 8 + 22 + 46 = 78.$$

Alſo $2 \times 8 + 2 \times 22 + 2.46 + 8 \times 22 + 8 \times 46 + 22 \times 46 = 1708$

And $2 \times 8 \times 22 + 2 \times 22 \times 46 + 2 \times 8 \times 46 + 8 \times 22 \times 46 = 11208.$

Alſo $2 \times 8 \times 22 \times 46 = 16192.$

If the equation is of either of the orders admitting more than one root, and one root is known, the remaining root or roots may be diſcovered by an equation of the third claſs.

The relations between the coparts and the roots of theſe equations may be found in the ſame manner as in the equations of the lower claſſes.

When the equation has more than one root, ſome number between any two roots being ſubſtituted for x, will make the unknown ſide greater than any other between thoſe two roots can do; and to find this, make $x = M + y$ or $M - z$ as in pages 29—67: and, by comparing together

together in the ſame manner the two equations thus formed, an equation will be diſcovered, whoſe roots are the different values of M, or the limits between the roots of the propoſed equation. Thus let the equation be of the form

$$x^4 - a x^3 + b x^2 - c x = k$$

The limiting equation is

$$4 x^3 - 3 a x^2 + 2 b x = c$$

or each term is to be multiplied into the index of the unknown term in it, and the marks of all the unknown terms are to remain the ſame except that of the laſt unknown term, which is made equal to the ſum or difference of the other unknown terms thus found, and then the new equation is divided by x. Let

$$c x - b x^2 - a x^3 - x^4 = k.$$

Then for the limiting equation

$$c x - 2 b x^2 - 3 a x^3 = 4 x^4$$

$$\therefore c - 2 b x - 3 a x^2 = 4 x^3.$$

Inſtance.

Let $11208 x - 1708 x^2 + 78 x^3 - x^4 = 16192.$

The limiting equation is

$$11208 x - 2.1708 x^2 + 3.78 x^3 = 4 x^4$$

or

$$11208 - 2.1708 x + 3.78 x^2 = 4 x^3$$

$$\therefore 2802 = 854 x - 58,5 x^2 + x^3$$

x	=	5	4	10	15	16
$\frac{2802}{x}$	=	560,4	700	280,2	186	175
$854 - 58,5 x + x^2$	=	586,5	636	369	202,5	174.

 Hence

Hence the three roots of the limiting equation are 4, ... 16, ... and 37, ... and consequently the roots of the proposed equation are the least, less than four, the next between 4, ... and 16, ... the third between 16, ... and 37, ... and the fourth between 37, ... and 57. The last number 57 is found by taking 4, ... + 16, ... from 78; for if two of the roots are greater than 4, ... + 16, ... that is than 20, ... the other two roots must be less than 78 — 20, ... that is than 57, ... and consequently the greatest root must be less than 57. In this case the limiting equation is of little or rather of no use towards the discovery of the roots of the proposed equation, as the four roots of the proposed equation are discovered with as little difficulty as these limiting roots.

The modes adopted for discovering the relation between the coparts and roots of equations of the second and third classes may be applied to this class, as may also the various methods for determining the number of roots in any order of this class. But from the greater number of unknown terms the trouble is rather increased.

Instances.

$$x^4 - 20x^3 + 9x^2 + 100x = 9246.$$

If x is less than 20, the unknown term is less than $9x^2 + 100x$, and if x is equal to 20 the unknown term is less than 9246, consequently the equation has only one root; for by adding to x, when the unknown term is equal to

9246,

9246, it is increafed, and by taking away from x the unknown term is diminifhed.

$$14937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000.$$

One root is evidently lefs than unity, and the unknown fide may be divided into two parts, of which the one $14937\,x - 1998\,x^2$ increafes till x is equal to $\frac{14937}{2.1998}$ or 3, ... and then decreafes till x is equal to $\frac{14937}{1998}$ or 7, ... The other part $80\,x^3 - x^4$ increafes till x is equal to 60, and then decreafes. Confequently, fince one root is lefs than unity there cannot be another root lefs than 3, ... Nor can there be another root between 3, ... and 7, ... for if $x = 4$, then $80\,x^3 - x^4$ is greater than 5000, and confequently the unknown fide muft be greater than 5000 when x is between 3, ... and 7, ... When x is greater than 7, ... then the part $14937\,x - 1998\,x^2$ is not to be added to $80\,x^3 - x^4$, but $1998\,x^2 - 14937\,x$ is to be taken away from it, and the increafe of $1998\,x^2 - 14937\,x$ is when x is 8 or a number greater than 8 confiderably greater than the increafe of $80\,x^3 - x^4$. Confequently there will be another root to this equation; but when $x = 20$ the increafe of $80\,x^3 - x^4$ is nearly equal to that of $1998\,x^2 - 14937\,x$, and when x is 30 the increafe of $80\,x^3 - x^4$ is much greater than that of $1998\,x^2 - 14937\,x$, confequently there will be a root between 7, ... and 20. Now the part $1998\,x^2 - 14937\,x$ conftantly increafes; but there is no increafe of $80\,x^3 - x^4$ when $x = 60$, confequently when x is between 30 and 60 the unknown fide muft become again equal to 5000,

5000, and unless this takes place when the unknown side is the greatest possible there must be two more roots to the equation *. But it is evident that the number making the unknown side the greatest possible is fractional, and therefore there will be four roots to the equation.

Call $14937 - 1998\,x + 80\,x^2 - x^3$ equal to z. Then $x \times z = 5000$

$$x = 10 \quad 12 \quad 13 \quad 30$$
$$\frac{5000}{x} = 500 \quad 416 \quad 384 \quad 166$$
$$z = 1957 \quad 753 \quad 286 \quad 3$$
$$x = 32 \quad 34 \quad 35$$
$$\frac{5000}{x} = 156 \quad 147 \quad 142$$
$$z = 153 \quad 181 \quad 132.$$

Hence the four roots are the one less than unity, the second between 12 and 13, the third between 32 and 33, but very nearly equal to 32, the fourth between 34 and 35, but much nearer to the latter than the former number.

Dr. Wallis and Mr. Raphson have employed themselves upon this equation; the former giving for one root 12,75644179448074402 . . . . the latter in his nine-

* The equation for finding the numbers which make the unknown side the greatest is

$$14937 = 3996\,x - 240\,x^2 + 4\,x^3.$$

In this equation x cannot be a whole number, for whether x is made an odd or an even number the unknown side is an even number, and consequently not equal to 14937.

teenth

teenth problem making this root 12,75644179448111, and Baron Maſeres, who is now employed upon the ſame equation, confirms * the ſolution given by Dr. Wallis.

The progreſs in approaching to theſe roots is as follows: The firſt root is evidently leſs than unity. 10 is both the eaſieſt and moſt obvious number to be tried for the next root which led to the trial of 12 and 13. Hence, two of the roots were found to be the one leſs than unity, and the other between 12 and 13. Conſequently the ſum of the remaining roots being equal to the difference between 80 and the ſum of the roots thus found is 67, ... nearly, the half of which being 33, ... is greater than one and leſs than the other remaining root. Hence 30 was tried, which led to the trial of 32, and 32 being found to be very nearly equal to one root, the other is neceſſarily very nearly equal to 34.

$$x^5 - 7x^4 + 20x^3 - 155x^2 = 10000.$$

The unknown ſide of this equation may be divided into two parts $x^5 - 7x^4$ and $20x^3 - 155x^2$. If x is leſs than 7, the unknown ſide is leſs than $20x^3 - 155x^2$. Alſo ſince $20x^3 - 155x^2 = x^2 \times \overline{20x - 155}$, if x is leſs than $\frac{155}{20}$ or 7,75, the unknown ſide is leſs than $x^5 - 7x^4$; and if when $x = 7{,}75$, the term $x^5 - 7x^4$ is leſs than 10000, the equation can have only one root, for when x is greater than 7,75 both parts $x^5 - 7x^4$ and

* In an Appendix, now in the preſs, to Halley's new, exact, and eaſie Method of finding the Roots of any Equations generally.

$20\,x^3 - 155\,x$ increaſe by adding $\dot{x}$ to x, and decreaſe by taking $\dot{x}$ from x. Let $x = 7{,}75$, then the unknown ſide is equal to $x^4 \times \overline{x - 7}$ or $x^4 \times \overline{7{,}75 - 7}$ or $,75 \times \overline{7{,}75}|^4$, that is, it is leſs than $,75 \times 8^4$ or $,75 \times 4096$, and conſequently is leſs than 10000. Therefore this equation has only one root.

The ſame thing may be proved by the limiting equation, which is

$$5\,x^4 - 28\,x^3 + 60\,x^2 = 310$$
$$\text{or } x^4 - 5{,}6\,x^3 + 12\,x^2 = 62.$$

It is evident, that if x is equal to or greater than 5,6, the unknown ſide is greater than 62, conſequently the roots of this equation muſt, if there are more than one, be each leſs than 5,6. But 5,6, or a number leſs than 5,6 cannot be a root of the equation.

$$x^5 - 7\,x^4 + 20\,x^3 - 155\,x^2 = 10000$$

which can therefore have but one root.

This equation is ſolved by Baron Maſeres in the third volume of the Scriptores Logarithmici, page 742, and the root determined to be very nearly equal to 8,6 or 8,5992325.

CHAP.

CHAP. VI.

GENERAL OBSERVATIONS.

BY the method purſued in the preceding chapters the relation between the coparts and the roots of any equation whatſoever will be diſcovered, and the equations of one mode in the laſt order of each claſs follow an analogy, which may eaſily be traced from the ſecond order of equations of the ſecond claſs to the laſt order of every claſs. This mode is when the difference between every two adjoining indexes is unity, and the higheſt index is the number which denominates the claſs to which the equation belongs. Thus $a x^n - x^m = k$ is the ſecond order of equations of the ſecond claſs; let $m = 2$ and $n = 1$, then a is equal to the ſum of the roots and k is equal to the product of the roots. $x^3 - a x^2 + b x = k$ is of the third order of the third claſs; the higheſt index is the number denoting the claſs, and the adjoining indexes differ by unity. a is equal to the ſum of the roots; b is equal to the ſum of the product of each pair of roots, and k is equal to the product of the roots. $c x - b x^2 + a x^3 - x^4 = k$. Here the equation is of the laſt order of the fourth claſs, the higheſt index is the number denoting the claſs, and the adjoining indexes differ by unity. a is equal to the ſum of the roots; b the ſum of the products of each pair of roots; c the ſum of the products of the roots taken by threes, and k the product of all the roots. Let n denote the higheſt index, and let the adjoining indexes differ by unity, then the equations of the laſt order of a claſs belong to one or

other of these forms according as the class is denoted by an even or an odd number

$$l\,x - \ldots + \ldots - \ldots + \ldots - d\,x^{n-4} + c\,x^{n-3} - b\,x^{n-2} + a\,x^{n-1} - x^n = k$$

$$x^n - a\,x^{n-1} + b\,x^{n-2} - c\,x^{n-3} + d\,x^{n-4} - e\,x^{n-5} \ldots = k.$$

In both cases, if the equation has n roots, a is equal to the sum of the roots, b is equal to the sum of the products of each pair of roots, c is equal to the sum of the products of the roots taken by threes, d is equal to the sum of the products of the roots taken by fours, and so on, and k is the product of all the roots.

The following is a general form to which every proposed equation may be made to correspond, observing only that no equation whatever can have all the lowest marks together:

$$\pm x^m \pm a\,x^n \pm b\,x^o \pm c\,x^p \pm d\,x^q \pm e\,x^r \ldots = k.$$

The limiting equation to the above equation is

$$\pm m\,x^{m-1} \pm n\,a\,x^{n-1} \pm o\,b\,x^{o-1} \pm p\,c\,x^{p-1} \ldots = \pm b\,x^{t-1} \pm v\,x^{v-1}.$$

The proof is the same as that adopted in pages 26, 27, 29, 67, 69.

It has been already observed, that this limiting equation will be of little use in equations of high classes. As an instance, let it be required to find the number of roots in the equation $x^8 + 4\,x^7 - x^6 - 10\,x^5 + 5\,x^4 - 5\,x^3 - 10\,x^2 - 10\,x = 5$.

The

The limiting equation is
$$8x^7+28x^6-6x^5-50x^4+20x^3-15x^2-20x=10$$

In this caſe the limiting equation is of a form capable of having as many roots as the propoſed equation, and it becomes now a queſtion to find the number of roots in the limiting equation. Inſtead of this, let the method adopted in pages 84, 85, 86, be applied to the propoſed equation.

$$x^8+4x^7-x^6-10x^5+5x^4-5x^3-10x^2-10x=5.$$

The unknown ſide may be divided into two parts $x^8 + 4x^7 - x^6 - 10x^5$ and $5x^4 - 5x^3 - 10x^2 - 10x$, and the equation is of a form capable of having three roots. Now the part $5x^4 - 5x^3 - 10x^2 - 10x$ will, if $5x^4$ is greater than the other terms, make a number to be added to the firſt part, and if it is leſs, make a number to be taken from the firſt part. The ſame may be obſerved of the firſt part if $x^8 + 4x^7$ is greater or leſs than $x^6 + 10x^5$. Now x muſt evidently be greater than unity, and $5x^4 - 5x^3 - 10x^2 - 10x$ makes a number to be taken from the firſt part if x is not greater than 2. But when x is equal to 2, or ſome number leſs than 2, $x^8 + 4x^7$ is greater than $x^6 + 10x^5$. Conſequently ſome number leſs than 2 will make $x^8 + 4x^7$ equal to $x^6 + 10x^5$, and then every addition to x increaſes the part $x^8 + 4x^7 - x^6 - 10x^5$. Now if the part $10x + 10x^2 + 5x^3 - 5x^4$ decreaſes by the addition of x to x the whole unknown ſide muſt neceſſarily increaſe. The part $10x + 10x^2 + 5x^3 - 5x^4$ firſt increaſes and then decreaſes, and the limit of its increaſe is when

$$10 = 20x^3 - 15x^2 - 20x$$

$$\text{or } ,5 = x^3 - ,75x^2 - x$$

that is x is not equal to 1,2, and consequently when x is greater than 1,2 the part $10x + 10x^2 + 5x^3 - 5x^4$ constantly diminishes till $10x + 10x^2 + 5x^3$ is equal to $5x^4$, and then by adding $\dot{x}$ to x as before, the part $5x^4 - 5x^3 - 10x^2 - 10x$ constantly increases. But when x is equal to 1,2 the unknown side of the given equation is no number at all, and consequently when the unknown side by x being a greater number, is equal to 5 the part $x^8 + 4x^7 - x^6 - 10x^5$ is increased by the addition of $\dot{x}$ to x, and the part to be taken from it is either less than it was before, and consequently the unknown side will be greater than 5, or instead of a number to be taken away from $x^8 + 4x^7 - x^6 - 10x^5$ the other part is a number to be added to it. Therefore the unknown side is always increased by adding $\dot{x}$ to x, or diminished by taking away $\dot{x}$ from x and the equation can have only one root. This root is discovered by Baron Maseres, in the Scriptores Logarithmici, Vol. III. p. 749, to be the only root, and to be very nearly equal to 1,618, and the equation is the highest numeral equation mentioned by Newton in his Arithmetica Universalis.

Let M be the number denoting the class, and N the number denoting an order in that class, then the number of orders in any class is equal to M, the number of forms in any class M is equal to $2^M - 1$, and the number of forms in any order N is equal to $M.\ \frac{M-1}{2}.\ \frac{M-2}{3}.\ \frac{M-3}{4} \ldots\ldots \frac{M-N+1}{N}$.

Thus,

Thus, let M be equal to 3, then in the third claſs there are three orders. Let M be equal to 100, and in the 100th claſs there are 100 orders.

Let M be equal to 4, then in the fourth claſs the number of forms is equal to $2^4 - 1$ or $16 - 1$ or 15.

Let $M = 4$ and $N = 3$, then the number of forms in the fourth claſs, capable of having three roots, is equal to $4 \times \frac{3}{2} \times \frac{2}{3}$ or 4. That is, there are four forms being capable of having three roots.

Let $M = 4$ and $N = 2$, then the number of forms in the fourth claſs, capable of having two roots, is $4 \times \frac{3}{2}$ or 6. That is, there are ſix forms capable of having two roots.

If the unknown term, having the higheſt index in an equation, is to be taken away from the reſt, the number of the order to which it belongs is an even number, and conſequently the greateſt number of roots an equation of that form can have is an even number; but if the unknown term, having the higheſt index, is not to be taken away from the reſt, then the order to which it belongs is an odd number, and conſequently the greateſt number of roots an equation of that form can have is an odd number.

Let N be the number denoting the order to which an equation belongs, then the number of the changes of the marks is equal to N or $N - 1$. Hence, by ob-

observing the changes of the marks in any equation, the greatest number of roots which an equation of that form is capable of having is ascertained. Let the number of changes be P, then, if the term with the highest index is the first term, N is an odd number, otherwise N is an even number. If P then is an odd number, and N an even number, P is equal to N — 1 or N is equal to P + 1; but if both P and N are even numbers or odd numbers then P is equal to N.

$$\text{Let } x^3 - 16\,x^2 + 77\,x = 98.$$

x^3 being the first term, the order to which this equation belongs is denoted by an odd number, but there is an even number of changes, therefore N = P + 1 or 3; that is, this equation cannot have more than three roots, and it depends upon the coparts and known term, whether it has one, two, or three roots.

$$\text{Let } x^8 + 4\,x^7 - x^6 - 10\,x^3 + 5\,x^4 - 5\,x^3 - 10\,x^2 - 10\,x = 5:$$

x^8 being the first term, the equation belongs to an order whose index is an odd number, and the number of changes in the marks is odd, therefore the order to which this equation belongs is the third; this equation cannot have more than three roots. If the equation had been

$$x^8 - 4\,x^7 + x^6 - 10\,x^3 + 5\,x^4 - 5\,x^3 + 10\,x^2 - 10\,x = 5,$$

by counting the unknown terms it appears that it belongs to the eighth class, and consequently cannot have more than eight roots: since x^8 is the first term, the number denoting the order is an odd number, and there being seven,

ſeven, that is an odd number of changes in the marks, the equation belongs to the ſeventh order, and cannot conſequently have more than ſeven roots.

The inveſtigation of the properties of equations is endleſs; it is with them as with intelligent beings. There is no limit to the number of modes of each form. There is no limit to the number of forms. There is no limit to the number of orders. There is no limit to the number of claſſes. Each mode has its peculiar curve. The lives of men of the firſt talents have been employed upon a ſingle curve, and there are not names given to a hundred ſpecies of curves. By the claſs of intelligent beings next in rank above man, all theſe equations and all theſe curves are, perhaps, thoroughly underſtood, and the next claſs excells them as much as they do us. How great then muſt be that being to whom the thoughts of all theſe orders of beings are known at a moment's glance; and how inſignificant in the eye of reaſon are thoſe nations which lay down rules for thought and perſecute for opinions.

THE END.

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